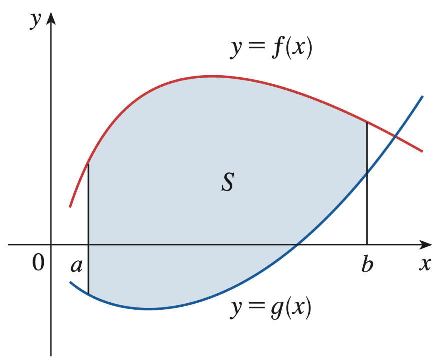
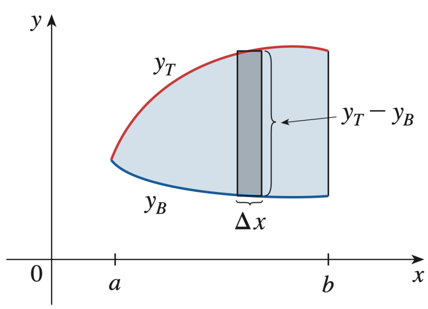
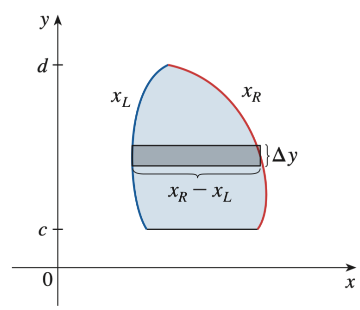
6.1 Areas Between Curves

1. Visualize the Problem

- Sketch the curves to understand their relative positions.
- Identify the region whose area needs to be calculated.



2. Decide the Variable of Integration

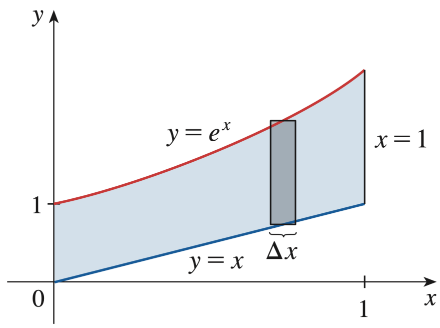
Integration with Respect to x	**Integration with Respect to y**
Choose to integrate with respect to x if the region is vertically bounded (i.e., top and bottom curves).  The area is given by: $$\text{Area} = \int_a^b \left(y_T - y_B\right) dx$$	Choose to integrate with respect to y if the region is horizontally bounded (i.e., right and left curves).  The area is given by: $$\text{Area} = \int_c^d \left(x_R - x_L\right) dy$$

3. Find the Limits of Integration

- These are either given, or they correspond to points of intersection.
- If needed, solve for the intersection points of the curves by setting them equal.

Area Between Curves: Integrating With Respect to x

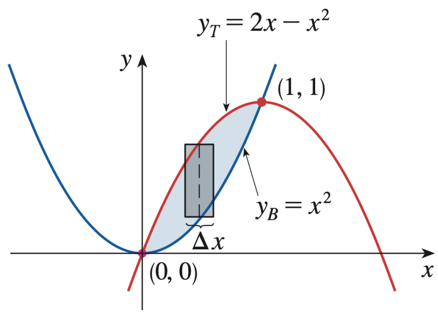
Example. Find the area of the region bounded above by $y = e^x$, bounded below by $y = x$, and bounded on the sides by $x = 0$ and $x = 1$.



$y_T = e^x$ and $y_B = x$

$$A = \int_a^b y_T - y_B \, dx = \int_0^1 e^x - x \, dx = e - 1.5$$

Example. Find the area of the region enclosed by the parabolas $y = x^2$ and $y = 2x - x^2$.



$y_T = 2x - x^2$ and $y_B = x^2$

Solve: $x^2 = 2x - x^2 \Rightarrow 2x^2 - 2x = 0$

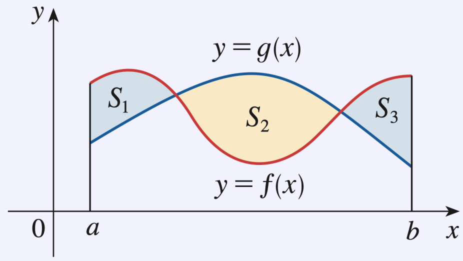
$\Rightarrow x^2 - x = 0$

$\Rightarrow x(x-1) = 0$

$\Rightarrow x = 0$ or $x = 1$

$$A = \int_0^1 \underbrace{(2x - x^2)}_{y_T} - \underbrace{x^2}_{y_B} \, dx = \int_0^1 2x - 2x^2 \, dx = \frac{1}{3}$$

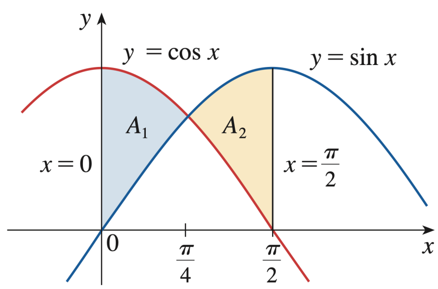
Sometimes we are asked to find the area between the curves $y = f(x)$ and and $y = g(x)$ where $f(x) \geq g(x)$ for some values of x but $g(x) \geq f(x)$ for other values of x.



In this case, we split the given region S into several regions $S_1, S_2, \ldots$ with areas $A_1, A_2, \ldots$. The total area is

$$A = A_1 + A_2 + \cdots = \int_a^b |f(x) - g(x)|\, dx$$

Example. Find the area of the region bounded by the curves $y = \sin x$, $y = \cos x$, $x = 0$, and $x = \pi/2$.



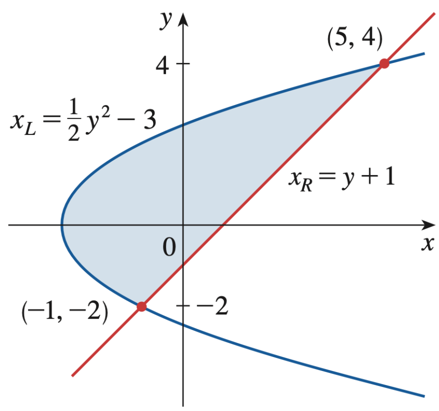
The point of intersection is when $\sin x = \cos x \Rightarrow x = \frac{\pi}{4}$

$$A = A_1 + A_2 = \int_0^{\pi/4} \overset{y_T}{\cos x} - \overset{y_B}{\sin x}\, dx + \int_{\pi/4}^{\pi/2} \overset{y_T}{\sin x} - \overset{y_B}{\cos x}\, dx$$

$$= 2\sqrt{2} - 2$$

Area Between Curves: Integrating With Respect to y

Example. Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.



$x_L = \frac{1}{2}y^2 - 3$ $\qquad$ $x_R = y+1$

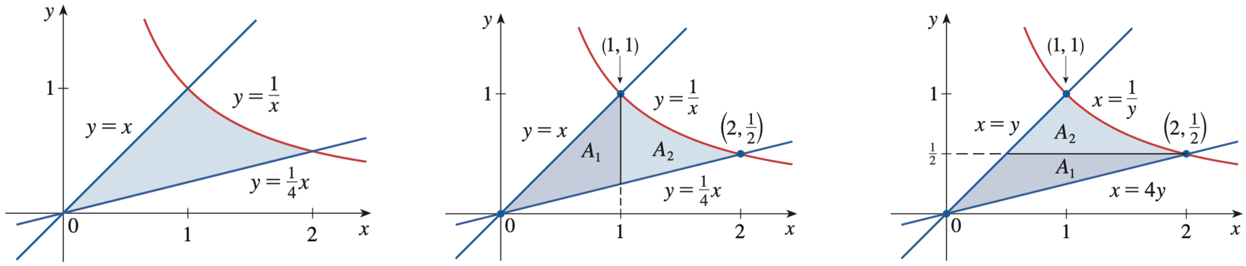
The intersection points are $(-1,-2)$ and $(5,4)$ [Solve]

$$A = \int_c^d x_R - x_L \, dy = \int_{-2}^{4} (y+1) - \left(\tfrac{1}{2}y^2 - 3\right) dy = 18$$

Example. Find the area of the region enclosed by the curves $y = 1/x$, $y = x$, and $y = \frac{1}{4}x$, using

(a) x as the variable of integration.

(b) y as the variable of integration.



(a) $A = A_1 + A_2 = \int_0^1 \overset{y_T}{x} - \overset{y_B}{\tfrac{1}{4}x}\,dx + \int_1^2 \overset{y_T}{\tfrac{1}{x}} - \overset{y_B}{\tfrac{1}{4}x}\,dx = \ln 2$

(b) $A = A_1 + A_2 = \int_0^{1/2} \overset{x_R}{4y} - \overset{x_L}{y}\,dy + \int_{1/2}^{1} \overset{x_R}{\tfrac{1}{y}} - \overset{x_L}{y}\,dy = \ln 2$