5.5 The Substitution Rule

Introduction

In calculus, many integrals involve composite functions that make direct integration challenging. The substitution rule provides a way to simplify such integrals by rewriting them in terms of a new variable. This process is analogous to "reversing" the chain rule of differentiation.

Question. How can we evaluate something like $\int 2x\sqrt{1+x^2}\,dx$?

Theorem (The Substitution Rule)**.** If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then:

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

Proof. Let F be an antiderivative of f, so that $F'(u) = f(u)$. Consider the substitution $u = g(x)$ and $du = g'(x)\,dx$:

$$\begin{aligned}
\int f(u)\,du &= F(u) + C \\
&= F(g(x)) + C \\
&= \int \frac{d}{dx} F(g(x))\,dx \\
&= \int F'(g(x))g'(x)\,dx \\
&= \int f(g(x))g'(x)\,dx
\end{aligned}$$

□

Example. Find $\int x^3 \cos(x^4+2)\,dx$.

Example. Evaluate $\int \sqrt{2x+1}\,dx$.

Example. Find $\int \frac{x}{\sqrt{1-4x^2}}\,dx$.

Example. Calculate $\int e^{5x}\,dx$.

Example. Calculate $\int \tan x \, dx$.

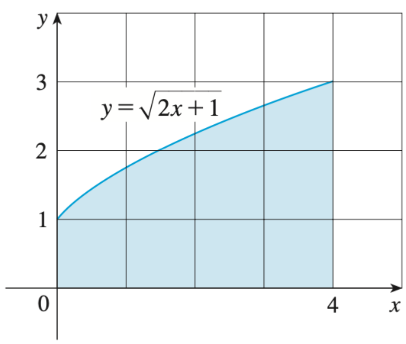
Definite Integrals

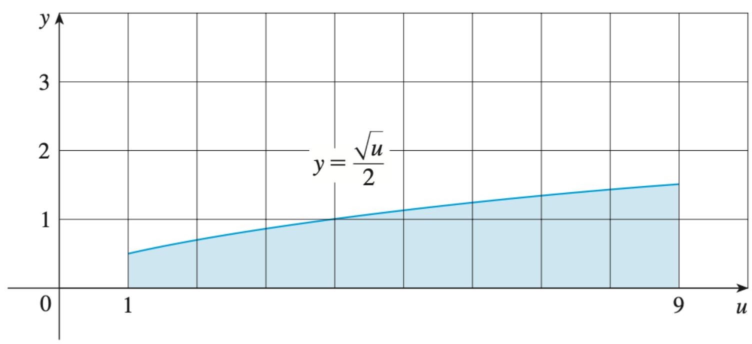
Question. What are two possible methods to evaluate a definite integral by substitution?

Theorem (The Substitution Rule for Definite Integrals)**.** If $g'(x)$ is continuous on the interval $[a, b]$ and f is continuous on the range of $u = g(x)$, then:

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Example. Evaluate $\int_0^4 \sqrt{2x+1}\,dx$.





Example. Evaluate $\int_1^2 \dfrac{dx}{(3-5x)^2}$.

Example. Calculate $\int_1^e \frac{\ln x}{x}\, dx$.