5.5 u-Substitution (Solutions)

1. $\int x\sqrt{1-x^2}\,dx$

   Substitution: Let $u = 1 - x^2$. Then $du = -2x\,dx \implies -\frac{1}{2}du = x\,dx$.

$$\begin{aligned}\int \sqrt{1-x^2}\cdot(x\,dx) &= \int u^{1/2}\cdot\left(-\frac{1}{2}\,du\right)\\ &= -\frac{1}{2}\int u^{1/2}\,du\\ &= -\frac{1}{2}\left(\frac{2}{3}u^{3/2}\right)+C\\ &= \boxed{-\frac{1}{3}(1-x^2)^{3/2}+C}\end{aligned}$$

2. $\int x\sec^2(x^2)\,dx$

   Substitution: Let $u = x^2$. Then $du = 2x\,dx \implies \frac{1}{2}du = x\,dx$.

$$\begin{aligned}\int \sec^2(x^2)\cdot(x\,dx) &= \int \sec^2(u)\cdot\left(\frac{1}{2}\,du\right)\\ &= \frac{1}{2}\int \sec^2(u)\,du\\ &= \frac{1}{2}\tan(u)+C\\ &= \boxed{\frac{1}{2}\tan(x^2)+C}\end{aligned}$$

3. $\int \frac{e^x}{1+e^x}\,dx$

   Substitution: Let $u = 1 + e^x$. Then $du = e^x\,dx$.

$$\begin{aligned}\int \frac{1}{1+e^x}\cdot(e^x\,dx) &= \int \frac{1}{u}\,du\\ &= \ln|u|+C\\ &= \boxed{\ln(1+e^x)+C}\end{aligned}$$

4. $\int \frac{x}{(x^2+1)^3}\,dx$

   Substitution: Let $u = x^2 + 1$. Then $du = 2x\,dx \implies \frac{1}{2}du = x\,dx$.

$$\begin{aligned}\int (x^2+1)^{-3}\cdot(x\,dx) &= \int u^{-3}\cdot\left(\frac{1}{2}\,du\right)\\ &= \frac{1}{2}\left(\frac{u^{-2}}{-2}\right)+C\\ &= -\frac{1}{4}u^{-2}+C\\ &= \boxed{-\frac{1}{4(x^2+1)^2}+C}\end{aligned}$$

5. $\displaystyle\int \frac{x}{1+x^4}\,dx$

Strategy: Rewrite x^4 *as* $(x^2)^2$.

Substitution: Let $u = x^2$. Then $du = 2x\,dx \implies \frac{1}{2}du = x\,dx$.

$$\begin{aligned}\int \frac{1}{1+(x^2)^2}\cdot(x\,dx) &= \int \frac{1}{1+u^2}\cdot\left(\frac{1}{2}\,du\right)\\ &= \frac{1}{2}\arctan(u) + C\\ &= \boxed{\frac{1}{2}\arctan(x^2) + C}\end{aligned}$$

6. $\displaystyle\int \frac{x}{\sqrt{x+2}}\,dx$

Substitution: Let $u = x+2$. Then $du = dx$.

Critical Step: Solve for $x \implies x = u-2$.

$$\begin{aligned}\int \frac{u-2}{\sqrt{u}}\,du &= \int \left(\frac{u}{u^{1/2}} - \frac{2}{u^{1/2}}\right)du\\ &= \int \left(u^{1/2} - 2u^{-1/2}\right)du\\ &= \left(\frac{2}{3}u^{3/2}\right) - 2\left(2u^{1/2}\right) + C\\ &= \boxed{\frac{2}{3}(x+2)^{3/2} - 4\sqrt{x+2} + C}\end{aligned}$$

7. $\displaystyle\int_0^1 (5x+1)^3\,dx$

Substitution: Let $u = 5x+1 \implies \frac{1}{5}du = dx$.

New Bounds: $x = 0 \to u = 1$; $x = 1 \to u = 6$.

$$\begin{aligned}\int_1^6 u^3\cdot\left(\frac{1}{5}\,du\right) &= \frac{1}{5}\left[\frac{u^4}{4}\right]_1^6\\ &= \frac{1}{20}(6^4 - 1^4)\\ &= \frac{1295}{20} = \boxed{\frac{259}{4}}\end{aligned}$$

8. $\displaystyle\int_0^{\pi/4} \sec^2(x)\tan(x)\,dx$

Substitution: Let $u = \tan(x) \implies du = \sec^2(x)\,dx$.

New Bounds: $x = 0 \to u = 0$; $x = \pi/4 \to u = 1$.

$$\begin{aligned}\int_0^1 u\,du &= \left[\frac{u^2}{2}\right]_0^1\\ &= \frac{1}{2}(1)^2 - \frac{1}{2}(0)^2\\ &= \boxed{\frac{1}{2}}\end{aligned}$$

9. $\int_0^{\sqrt{7}} x\sqrt{x^2+1}\,dx$

Substitution: Let $u = x^2 + 1 \implies \frac{1}{2}du = x\,dx$.
New Bounds: $x = 0 \to u = 1$; $\quad x = \sqrt{7} \to u = 8$.

$$\begin{aligned}\int_1^8 u^{1/2} \cdot \left(\frac{1}{2}\,du\right) &= \frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_1^8 \\ &= \frac{1}{3}(8^{3/2} - 1^{3/2}) \\ &= \frac{1}{3}(16\sqrt{2} - 1) = \boxed{\frac{16\sqrt{2}-1}{3}}\end{aligned}$$

10. $\int_0^{\ln 2} e^{-x}\,dx$

Substitution: Let $u = -x \implies -du = dx$.
New Bounds: $x = 0 \to u = 0$; $\quad x = \ln 2 \to u = -\ln 2$.

$$\begin{aligned}\int_0^{-\ln 2} e^u(-du) &= -\left[e^u\right]_0^{-\ln 2} \\ &= -(e^{-\ln 2} - e^0) \\ &= -(\tfrac{1}{2} - 1) \\ &= \boxed{\frac{1}{2}}\end{aligned}$$