5.5 The Substitution Rule

Introduction

In calculus, many integrals involve composite functions that make direct integration challenging. The substitution rule provides a way to simplify such integrals by rewriting them in terms of a new variable. This process is analogous to "reversing" the chain rule of differentiation.

Question. How can we evaluate something like $\int 2x\sqrt{1+x^2}\,dx$?

To "undo" the chain rule, we need functions in the form $f(g(x)) \cdot g'(x)$

Consider: $f(u) = \sqrt{u}$, $g(x) = 1+x^2$, and $g'(x) = 2x$. Then:

$$f(g(x)) \cdot g'(x) = \sqrt{1+x^2} \cdot 2x$$

Let $u = g(x) = 1+x^2$. Then $du = g'(x)\,dx = 2x\,dx$. We get:

$$\int 2x \cdot \sqrt{1+x^2}\,dx = \int \sqrt{u}\,du$$

Theorem (The Substitution Rule)**.** If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then:

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

Proof. Let F be an antiderivative of f, so that $F'(u) = f(u)$. Consider the substitution $u = g(x)$ and $du = g'(x)\,dx$:

$$\begin{aligned}
\int f(u)\,du &= F(u) + C && \text{def. of integral} \\
&= F(g(x)) + C && u = g(x) \\
&= \int \frac{d}{dx} F(g(x))\,dx && \text{Fund. Thm. of Calc} \\
&= \int F'(g(x))g'(x)\,dx && \text{Chain rule} \\
&= \int f(g(x))g'(x)\,dx && F' = f
\end{aligned}$$

□

Example. Find $\int x^3 \cos(x^4+2)\,dx$.

Let $u = x^4+2$. Then $du = 4x^3\,dx \Rightarrow x^3 dx = \frac{1}{4} du$.

Substituting, we get

$$\int \cos(u) \cdot \frac{1}{4}\,du$$

Integrate:

$$\frac{1}{4}\sin(u) + C$$

Substitute back $u = x^4+2$

$$\boxed{\frac{1}{4}\sin(x^4+2) + C}$$

Example. Evaluate $\int \sqrt{2x+1}\,dx$.

Let $u = 2x+1$. Then $du = 2\,dx \Rightarrow dx = \frac{1}{2}du$. We get:

$$\begin{aligned} \int \sqrt{u} \cdot \frac{1}{2}\,du &= \frac{1}{2}\int u^{1/2}\,du \\ &= \frac{1}{2}\cdot\frac{2}{3}u^{3/2} + C \\ &= \frac{1}{3}u^{3/2} + C \\ &= \boxed{\frac{1}{3}(2x+1)^{3/2} + C} \end{aligned}$$

integrate

substitute back

Example. Find $\int \frac{x}{\sqrt{1-4x^2}}\,dx$.

Let $u = 1-4x^2$. Then $du = -8x\,dx \Rightarrow x\,dx = -\frac{1}{8}du$. We get:

$$\int \frac{1}{\sqrt{u}} \cdot -\frac{1}{8}\,du = -\frac{1}{8}\int u^{-1/2}\,du = -\frac{1}{8}\cdot 2u^{1/2} + C$$

$$= -\frac{1}{4}u^{1/2} + C$$

$$= \boxed{-\frac{1}{4}\sqrt{1-4x^2} + C}$$

Example. Calculate $\int e^{5x}\,dx$.

Let $u = 5x$. Then $du = 5\,dx \Rightarrow dx = \frac{1}{5}du$. We get:

$$\int e^u \cdot \frac{1}{5}\,du = \frac{1}{5}e^u + C$$

$$= \frac{1}{5}e^{5x} + C$$

Example. Calculate $\int \tan x \, dx$.

Recall: $\tan x = \dfrac{\sin x}{\cos x}$

Let $u = \cos x$. Then $du = -\sin x \, dx \Rightarrow \sin x \, dx = -du$

We get

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx = \int \frac{1}{u} \cdot -du = -\int \frac{1}{u} du$$

$$= -\ln|u| + C$$

$$= \boxed{-\ln|\cos x| + C}$$

Definite Integrals

Question. What are two possible methods to evaluate a definite integral by substitution?

1. Evaluate the indefinite integral first. Then plug in the bounds after substituting x back in for u.

2. Change the limits of integration to reflect the substitution (and keep things in terms of u)

Theorem (The Substitution Rule for Definite Integrals)**.** If $g'(x)$ is continuous on the interval $[a, b]$ and f is continuous on the range of $u = g(x)$, then:

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Example. Evaluate $\int_0^4 \sqrt{2x+1}\,dx$.

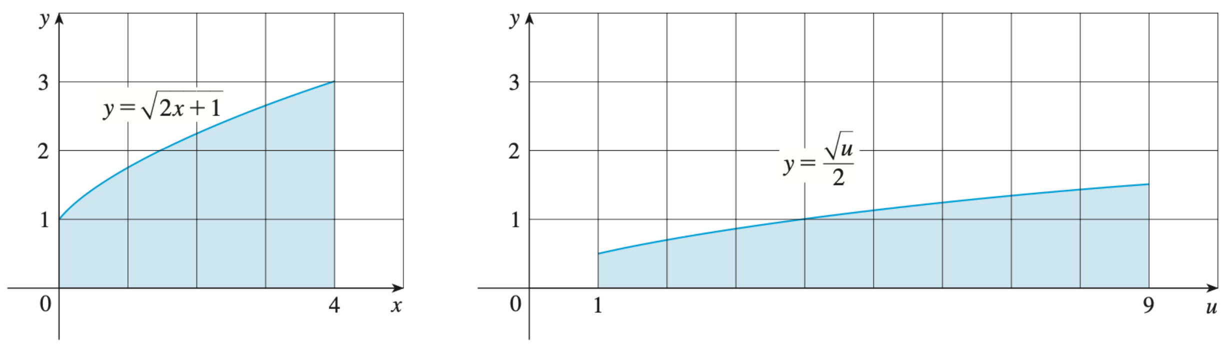
Let $u = 2x+1$. Then $du = 2\,dx \Rightarrow dx = \frac{1}{2}du$

Adjust the limits: when $x=0$, $u=1$. when $x=4$, $u=9$.

$$\int_0^4 \sqrt{2x+1}\,dx = \int_1^9 \sqrt{u}\cdot\frac{1}{2}du$$

$$= \frac{1}{3}\left[u^{3/2}\right]_{u=1}^{u=9}$$

$$= \frac{1}{3}\left[9^{3/2} - 1^{3/2}\right] = \frac{1}{3}(27-1) = \boxed{\frac{26}{3}}$$



Example. Evaluate $\int_1^2 \dfrac{dx}{(3-5x)^2}$.

Let $u = 3-5x$. Then $du = -5\,dx \Rightarrow dx = -\frac{1}{5}du$

Adjust the limits. When $x=1$, $u=-2$. when $x=2$, $u=-7$.

We get

$$\int_1^2 \frac{dx}{(3-5x)^2} = \int_{-2}^{-7} \frac{1}{u^2}\cdot -\frac{1}{5}du = -\frac{1}{5}\int_{-2}^{-7}\frac{1}{u^2}du$$

$$= -\frac{1}{5}\left[-\frac{1}{u}\right]_{u=-2}^{u=-7} = -\frac{1}{5}\left[-\frac{1}{-7}+\frac{1}{-2}\right] = -\frac{1}{5}\left[\frac{1}{7}-\frac{1}{2}\right]$$

$$= -\frac{1}{5}\cdot -\frac{5}{14} = \boxed{\frac{1}{14}}$$

Example. Calculate $\int_1^e \frac{\ln x}{x}\, dx$.

Let $u = \ln x$. Then $du = \frac{1}{x}\, dx$

Adjust the limits: when $x = 1$, $u = 0$. when $x = e$, $u = 1$.

We get:

$$\int_1^e \frac{\ln x}{x}\, dx = \int_0^1 u\, du$$

$$= \left[\frac{u^2}{2} \right]_{u=0}^{u=1}$$

$$= \frac{1^2}{2} - \frac{0^2}{2}$$

$$= \frac{1}{2}$$