11.9 Representations of Functions as Power Series

In this section we see how to represent some familiar functions as sums of power series.

- This is useful for integrating functions that don't have elementary antiderivatives and for approximating functions by polynomials.
- Scientists do this to simplify the expressions they deal with; computer scientists do this to evaluate functions on calculators and computers.

Power series representations of functions can be systematically derived by manipulating the geometric series formula. The process involves the following steps:

1. **Start with the geometric series formula:**

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n, \quad |x| < 1.$$

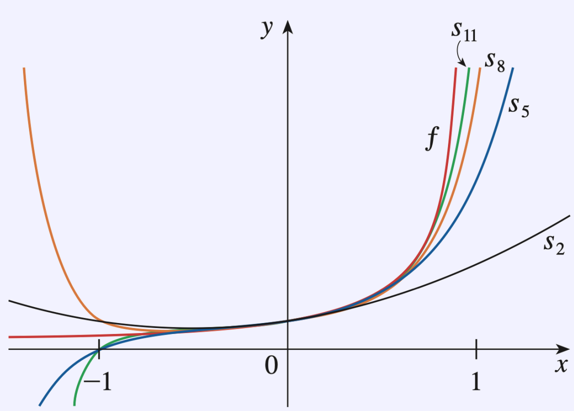
   This is a foundational representation for $f(x) = \dfrac{1}{1-x}$, valid on the interval $(-1, 1)$.

2. **Manipulate the geometric series to represent other functions:**
   - *Substitute variables:* Replace x with expressions like $-x$, x^2, or $x - a$ to adjust the series for related functions.
   - *Differentiate or integrate term-by-term:* Use differentiation or integration to derive series for functions such as $\ln(1+x)$, $\tan^{-1} x$, or similar.

3. **Understand the partial sums and convergence:** The sum of a power series is the limit of its sequence of partial sums. For $f(x) = \dfrac{1}{1-x}$,

$$s_n(x) = 1 + x + x^2 + \cdots + x^n \quad \text{and} \quad \frac{1}{1-x} = \lim_{n\to\infty} s_n(x).$$

   For $|x| < 1$, the partial sums $s_n(x)$ become increasingly accurate approximations of $f(x)$ as $n \to \infty$.



Example. Express $\dfrac{1}{1+x^2}$ as the sum of a power series and find the interval of convergence.

Example. Find a power series representation for $\dfrac{1}{x+2}$.

Example. Find a power series representation for $\dfrac{x^3}{x+2}$.

Theorem. If the power series $\sum c_n(x-a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and:

(i) $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \displaystyle\sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$

(ii) $\displaystyle\int f(x)\,dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots = C + \sum_{n=0}^{\infty} \frac{c_n(x-a)^{n+1}}{n+1}$

The radii of convergence of the power series in Equations (i) and (ii) are both R.

Remark. Equations (i) and (ii) can be rewritten in the form:

(iii) $\displaystyle\frac{d}{dx}\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] = \sum_{n=0}^{\infty} \frac{d}{dx}[c_n(x-a)^n]$

(iv) $\displaystyle\int \left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] dx = \sum_{n=0}^{\infty} \int c_n(x-a)^n dx$

We know that, for finite sums, the derivative of a sum is the sum of the derivatives and the integral of a sum is the sum of the integrals. The theorem says the same is true for infinite sums, provided we are dealing with *power series*.

Remark. Although the theorem says that the radius of convergence remains the same when a power series is differentiated or integrated, this does not mean that the *interval* of convergence remains the same. It may happen that the original series converges at an endpoint, whereas the differentiated series diverges there.

Example. Express $\dfrac{1}{(1-x)^2}$ as a power series. What is the radius of convergence?

Example. Find a power series representation for $\ln(1+x)$ and its radius of convergence.

Example. Find a power series representation for $f(x) = \tan^{-1} x$.

Example.

(a) Evaluate $\displaystyle\int \frac{1}{1+x^7}\,dx$ as a power series.

(b) Use part (a) to approximate $\displaystyle\int_0^{0.5} \frac{1}{1+x^7}\,dx$ correct to within 10^{-7}.