

11.8 Power Series (Solutions)

1.
$$\sum_{n=1}^{\infty} \frac{(2x-5)^n}{n \cdot 3^n}$$

Let

$$a_n = \frac{(2x-5)^n}{n \cdot 3^n}.$$

To find the radius and interval of convergence, we apply the Ratio Test:

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(2x-5)^{n+1}}{(n+1)3^{n+1}} \cdot \frac{n3^n}{(2x-5)^n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{2x-5}{3} \cdot \frac{n}{n+1} \right| \\ &= \left| \frac{2x-5}{3} \right|. \end{aligned}$$

The series converges when

$$\left| \frac{2x-5}{3} \right| < 1.$$

This inequality is equivalent to

$$|2x-5| < 3,$$

so

$$-3 < 2x-5 < 3.$$

Adding 5 throughout gives

$$2 < 2x < 8,$$

and dividing by 2 gives

$$1 < x < 4.$$

Thus the open interval of convergence is $(1, 4)$.

Now we test the endpoints.

At $x = 1$:

$$\sum_{n=1}^{\infty} \frac{(2(1)-5)^n}{n \cdot 3^n} = \sum_{n=1}^{\infty} \frac{(-3)^n}{n \cdot 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}.$$

This is an alternating harmonic series. It converges by the Alternating Series Test since

$$b_n = \frac{1}{n} > 0, \quad b_{n+1} < b_n, \quad \lim_{n \rightarrow \infty} b_n = 0.$$

At $x = 4$:

$$\sum_{n=1}^{\infty} \frac{(2(4)-5)^n}{n \cdot 3^n} = \sum_{n=1}^{\infty} \frac{3^n}{n \cdot 3^n} = \sum_{n=1}^{\infty} \frac{1}{n},$$

which is the harmonic series, so it diverges.

Therefore,

$$\boxed{R = \frac{3}{2}, \quad I = [1, 4].}$$

$$2. \sum_{n=1}^{\infty} \frac{n^2(x+4)^n}{7^n}$$

Let

$$a_n = \frac{n^2(x+4)^n}{7^n}.$$

Apply the Ratio Test:

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)^2(x+4)^{n+1}}{7^{n+1}} \cdot \frac{7^n}{n^2(x+4)^n} \right| \\ &= \lim_{n \rightarrow \infty} \left(\frac{(n+1)^2}{n^2} \cdot \left| \frac{x+4}{7} \right| \right) \\ &= \left| \frac{x+4}{7} \right|. \end{aligned}$$

The series converges when

$$\left| \frac{x+4}{7} \right| < 1.$$

This gives

$$|x+4| < 7,$$

so

$$-7 < x+4 < 7.$$

Subtracting 4 yields

$$-11 < x < 3.$$

Thus the open interval of convergence is $(-11, 3)$.

Now test the endpoints.

At $x = -11$:

$$\sum_{n=1}^{\infty} \frac{n^2(-11+4)^n}{7^n} = \sum_{n=1}^{\infty} \frac{n^2(-7)^n}{7^n} = \sum_{n=1}^{\infty} n^2(-1)^n.$$

Here the terms are

$$a_n = n^2(-1)^n.$$

Since

$$\lim_{n \rightarrow \infty} a_n$$

does not exist, and in particular the terms do *not* approach 0, the series diverges by the Test for Divergence.

At $x = 3$:

$$\sum_{n=1}^{\infty} \frac{n^2(3+4)^n}{7^n} = \sum_{n=1}^{\infty} \frac{n^2 7^n}{7^n} = \sum_{n=1}^{\infty} n^2.$$

Again, the terms do not approach 0, so the series diverges by the Test for Divergence.

Therefore,

$$\boxed{R = 7, \quad I = (-11, 3).}$$

$$3. \sum_{n=0}^{\infty} \frac{(x-1)^n}{n!}$$

Let

$$a_n = \frac{(x-1)^n}{n!}.$$

Apply the Ratio Test:

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(x-1)^{n+1}}{(n+1)!} \cdot \frac{n!}{(x-1)^n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{x-1}{n+1} \right| \\ &= 0.\end{aligned}$$

Since $0 < 1$ for every real number x , the series converges for all x .

Therefore, the radius of convergence is infinite, and the interval of convergence is all real numbers:

$$\boxed{R = \infty, \quad I = (-\infty, \infty).}$$

4.
$$\sum_{n=1}^{\infty} \frac{(x+2)^{2n}}{5^n}$$

Rewrite the series as

$$\sum_{n=1}^{\infty} \left(\frac{(x+2)^2}{5} \right)^n.$$

This is a geometric series with common ratio

$$r = \frac{(x+2)^2}{5}.$$

A geometric series converges exactly when $|r| < 1$. Therefore, we require

$$\left| \frac{(x+2)^2}{5} \right| < 1.$$

Since $(x+2)^2 \geq 0$, this becomes

$$\frac{(x+2)^2}{5} < 1,$$

so

$$(x+2)^2 < 5.$$

Taking square roots gives

$$-\sqrt{5} < x+2 < \sqrt{5},$$

and subtracting 2 gives

$$-2 - \sqrt{5} < x < -2 + \sqrt{5}.$$

Now test the endpoints.

At $x = -2 - \sqrt{5}$:

$$x+2 = -\sqrt{5}, \quad \frac{(x+2)^2}{5} = \frac{5}{5} = 1.$$

So the series becomes

$$\sum_{n=1}^{\infty} 1,$$

which diverges.

At $x = -2 + \sqrt{5}$:

$$x+2 = \sqrt{5}, \quad \frac{(x+2)^2}{5} = \frac{5}{5} = 1.$$

Again the series becomes

$$\sum_{n=1}^{\infty} 1,$$

which diverges.

Therefore,

$$\boxed{R = \sqrt{5}, \quad I = (-2 - \sqrt{5}, -2 + \sqrt{5}).}$$

5.
$$\sum_{n=1}^{\infty} \frac{(-1)^n (3x)^n}{n^2}$$

We may rewrite the series as

$$\sum_{n=1}^{\infty} \frac{(-3x)^n}{n^2}.$$

Let

$$a_n = \frac{(-3x)^n}{n^2}.$$

Apply the Ratio Test:

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(-3x)^{n+1}}{(n+1)^2} \cdot \frac{n^2}{(-3x)^n} \right| \\ &= \lim_{n \rightarrow \infty} \left(3|x| \cdot \frac{n^2}{(n+1)^2} \right) \\ &= 3|x|. \end{aligned}$$

The series converges when

$$3|x| < 1,$$

that is,

$$|x| < \frac{1}{3}.$$

So the open interval of convergence is

$$\left(-\frac{1}{3}, \frac{1}{3} \right).$$

Now test the endpoints.

At $x = \frac{1}{3}$:

$$\sum_{n=1}^{\infty} \frac{(-1)^n (3 \cdot \frac{1}{3})^n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}.$$

Since

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

is a p -series with $p = 2 > 1$, the series converges absolutely.

At $x = -\frac{1}{3}$:

$$\sum_{n=1}^{\infty} \frac{(-1)^n (3 \cdot (-\frac{1}{3}))^n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2},$$

which is also a p -series with $p = 2 > 1$. Hence it converges.

Therefore,

$$\boxed{R = \frac{1}{3}, \quad I = \left[-\frac{1}{3}, \frac{1}{3} \right].}$$

$$6. \sum_{n=1}^{\infty} \frac{x^n}{(n+2)!}$$

Let

$$a_n = \frac{x^n}{(n+2)!}.$$

To determine the radius and interval of convergence, we apply the Ratio Test:

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{(n+3)!} \cdot \frac{(n+2)!}{x^n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{x}{n+3} \right| \\ &= 0. \end{aligned}$$

Since $0 < 1$ for every real number x , the series converges for all real x .

Therefore,

$$\boxed{R = \infty, \quad I = (-\infty, \infty).}$$

$$7. \sum_{n=2}^{\infty} \frac{(x-7)^n}{n \ln n}$$

Let

$$a_n = \frac{(x-7)^n}{n \ln n}.$$

Apply the Ratio Test:

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(x-7)^{n+1}}{(n+1) \ln(n+1)} \cdot \frac{n \ln n}{(x-7)^n} \right| \\ &= |x-7| \lim_{n \rightarrow \infty} \frac{n}{n+1} \cdot \frac{\ln n}{\ln(n+1)} \\ &= |x-7|. \end{aligned}$$

Thus the series converges when

$$|x-7| < 1,$$

which gives

$$6 < x < 8.$$

So the open interval of convergence is $(6, 8)$, and the center is 7. Hence

$$R = 1.$$

Now test the endpoints.

At $x = 6$:

$$\sum_{n=2}^{\infty} \frac{(6-7)^n}{n \ln n} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}.$$

This is an alternating series of the form

$$\sum_{n=2}^{\infty} (-1)^n b_n \quad \text{with} \quad b_n = \frac{1}{n \ln n}.$$

To apply the Alternating Series Test, we verify the required conditions:

$$b_n > 0 \quad \text{for } n \geq 2,$$

since $n > 0$ and $\ln n > 0$ for $n \geq 2$.

Also, $b_n \rightarrow 0$ as $n \rightarrow \infty$ because $n \ln n \rightarrow \infty$.

Finally, b_n is decreasing for $x \geq 2$. Indeed, let

$$f(x) = \frac{1}{x \ln x}.$$

Then

$$f'(x) = -\frac{\ln x + 1}{x^2(\ln x)^2} < 0 \quad \text{for } x \geq 2.$$

So f is decreasing, which means $b_n = \frac{1}{n \ln n}$ is decreasing.

Therefore,

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$$

converges by the Alternating Series Test.

At $x = 8$:

$$\sum_{n=2}^{\infty} \frac{(8 - 7)^n}{n \ln n} = \sum_{n=2}^{\infty} \frac{1}{n \ln n}.$$

This is a positive-term series. Using the Integral Test with

$$f(x) = \frac{1}{x \ln x},$$

we know f is continuous, positive, and decreasing for $x \geq 2$, and

$$\int_2^{\infty} \frac{1}{x \ln x} dx = \lim_{t \rightarrow \infty} \int_2^t \frac{1}{x \ln x} dx.$$

Let $u = \ln x$, so $du = \frac{1}{x} dx$. Then

$$\int_2^t \frac{1}{x \ln x} dx = \int_{\ln 2}^{\ln t} \frac{1}{u} du = \ln(\ln t) - \ln(\ln 2).$$

As $t \rightarrow \infty$, this tends to ∞ . Hence

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

diverges.

Therefore,

$$\boxed{R = 1, \quad I = [6, 8].}$$

8. $\sum_{n=0}^{\infty} \frac{n!(x+1)^n}{n^n}$

Let

$$a_n = \frac{n!(x+1)^n}{n^n}.$$

Apply the Ratio Test:

$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{(n+1)!(x+1)^{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{n!(x+1)^n}\right| \\ &= |x+1| \cdot \frac{(n+1)n^n}{(n+1)^{n+1}} \\ &= |x+1| \left(\frac{n}{n+1}\right)^n.\end{aligned}$$

Now

$$\left(\frac{n}{n+1}\right)^n = \left(1 + \frac{1}{n}\right)^{-n} \rightarrow \frac{1}{e}.$$

Therefore,

$$\lim_{n \rightarrow \infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{|x+1|}{e}.$$

The series converges when

$$\frac{|x+1|}{e} < 1,$$

that is,

$$|x+1| < e.$$

So the open interval of convergence is

$$-1 - e < x < -1 + e.$$

Hence the center is -1 and the radius is

$$R = e.$$

Now test the endpoints.

At $x = -1 + e$:

$$x + 1 = e,$$

so the series becomes

$$\sum_{n=0}^{\infty} \frac{n!e^n}{n^n}.$$

Let

$$b_n = \frac{n!e^n}{n^n}.$$

Then

$$\frac{b_{n+1}}{b_n} = e \left(\frac{n}{n+1}\right)^n.$$

Since

$$\left(1 + \frac{1}{n}\right)^n < e,$$

taking reciprocals gives

$$\left(\frac{n}{n+1}\right)^n > \frac{1}{e}.$$

Therefore,

$$\frac{b_{n+1}}{b_n} = e \left(\frac{n}{n+1}\right)^n > 1.$$

So $\{b_n\}$ is increasing. Since $b_1 = e > 0$, it follows that $b_n \geq e$ for all $n \geq 1$. In particular,

$$\lim_{n \rightarrow \infty} b_n \neq 0.$$

Therefore, the series diverges by the Test for Divergence.

At $x = -1 - e$:

$$x + 1 = -e,$$

so the series becomes

$$\sum_{n=0}^{\infty} \frac{n!(-e)^n}{n^n}.$$

Its terms have absolute value

$$\frac{n!e^n}{n^n} = b_n.$$

From the work above, b_n is increasing and does not approach 0. Hence the terms of the series do not approach 0, so the series diverges by the Test for Divergence.

Therefore,

$$\boxed{R = e, \quad I = (-1 - e, -1 + e)}.$$

9.
$$\sum_{n=1}^{\infty} \frac{(2x - 3)^{2n}}{n \cdot 4^n}$$

Let

$$a_n = \frac{(2x - 3)^{2n}}{n \cdot 4^n}.$$

Apply the Ratio Test:

$$\begin{aligned} \left| \frac{a_{n+1}}{a_n} \right| &= \left| \frac{(2x - 3)^{2n+2}}{(n + 1)4^{n+1}} \cdot \frac{n4^n}{(2x - 3)^{2n}} \right| \\ &= \frac{(2x - 3)^2}{4} \cdot \frac{n}{n + 1}. \end{aligned}$$

Taking the limit gives

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{(2x - 3)^2}{4}.$$

The series converges when

$$\frac{(2x - 3)^2}{4} < 1.$$

This is equivalent to

$$(2x - 3)^2 < 4,$$

so

$$|2x - 3| < 2.$$

Therefore,

$$-2 < 2x - 3 < 2.$$

Adding 3 gives

$$1 < 2x < 5,$$

and dividing by 2 gives

$$\frac{1}{2} < x < \frac{5}{2}.$$

Thus the open interval of convergence is

$$\left(\frac{1}{2}, \frac{5}{2} \right).$$

The center is $x = \frac{3}{2}$, so the radius is

$$R = 1.$$

Now test the endpoints.

At $x = \frac{5}{2}$:

$$2x - 3 = 2,$$

so the series becomes

$$\sum_{n=1}^{\infty} \frac{2^{2n}}{n4^n} = \sum_{n=1}^{\infty} \frac{4^n}{n4^n} = \sum_{n=1}^{\infty} \frac{1}{n},$$

which diverges.

At $x = \frac{1}{2}$:

$$2x - 3 = -2,$$

so

$$(2x - 3)^{2n} = (-2)^{2n} = 4^n.$$

Thus the series again becomes

$$\sum_{n=1}^{\infty} \frac{1}{n},$$

which diverges.

Therefore,

$$\boxed{R = 1, \quad I = \left(\frac{1}{2}, \frac{5}{2}\right)}.$$

10. $\sum_{n=1}^{\infty} \frac{n(x-5)^n}{2^n(n+1)}$

Let

$$a_n = \frac{n(x-5)^n}{2^n(n+1)}.$$

Apply the Ratio Test:

$$\begin{aligned} \left| \frac{a_{n+1}}{a_n} \right| &= \left| \frac{(n+1)(x-5)^{n+1}}{2^{n+1}(n+2)} \cdot \frac{2^n(n+1)}{n(x-5)^n} \right| \\ &= \left| \frac{x-5}{2} \right| \cdot \frac{(n+1)^2}{n(n+2)}. \end{aligned}$$

Taking the limit gives

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{x-5}{2} \right|.$$

The series converges when

$$\left| \frac{x-5}{2} \right| < 1,$$

which is equivalent to

$$|x-5| < 2.$$

So

$$3 < x < 7.$$

Thus the open interval of convergence is $(3, 7)$, centered at $x = 5$, with radius

$$R = 2.$$

Now test the endpoints.

At $x = 7$:

$$\sum_{n=1}^{\infty} \frac{n(7-5)^n}{2^n(n+1)} = \sum_{n=1}^{\infty} \frac{n \cdot 2^n}{2^n(n+1)} = \sum_{n=1}^{\infty} \frac{n}{n+1}.$$

But

$$\lim_{n \rightarrow \infty} \frac{n}{n+1} = 1 \neq 0,$$

so the series diverges by the Test for Divergence.

At $x = 3$:

$$\sum_{n=1}^{\infty} \frac{n(3-5)^n}{2^n(n+1)} = \sum_{n=1}^{\infty} \frac{n(-2)^n}{2^n(n+1)} = \sum_{n=1}^{\infty} (-1)^n \frac{n}{n+1}.$$

Its terms are

$$(-1)^n \frac{n}{n+1}.$$

Since

$$\lim_{n \rightarrow \infty} \frac{n}{n+1} = 1,$$

the terms do not approach 0. Therefore, the series diverges by the Test for Divergence.

Hence,

$$\boxed{R = 2, \quad I = (3, 7)}.$$