11.8 Power Series

So far we have studied series of numbers: $\sum a_n$. Here we consider series, called *power series*, in which each term includes a power of the variable x: $\sum c_n x^n$.

Definition. A **power series** is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

Think: infinite polynomial

where x is a variable and the c_n's are constants called the **coefficients** of the series. For each number that we substitute for x, the series is a series of constants that we can test for convergence or divergence. A power series may converge for some values of x and diverge for other values of x. The sum of the series is a function

$$f(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$$

whose domain is the set of all x for which the series converges. Notice that f resembles a polynomial. The only difference is that f has infinitely many terms.

Example. If we take $c_n = 1$ for all n, the power series becomes the geometric series:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots$$

Determine the values of x for which this series converges and diverges.

For each value of x, we get a geometric series with common ratio x

This series converges when $-1 < x < 1$ and diverges when $|x| \geq 1$

For example: if $x = \frac{1}{2}$, $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 1 + \frac{1}{2} + \frac{1}{4} + \dots$ converges to 2

if $x = 2$, $\sum_{n=0}^{\infty} 2^n = 1 + 2 + 4 + \dots$ diverges

This power series converges to $\frac{a}{1-r} = \boxed{\frac{1}{1-x}}$

Definition. A series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

is called a **power series in** $(x-a)$, a **power series centered at** a, or a **power series about** a.

Notice that in writing out the term corresponding to $n = 0$, we adopt the convention that $(x-a)^0 = 1$ even when $x = a$. Notice also that when $x = a$, all of the terms are 0 for $n \geq 1$ and so the power series always converges when $x = a$.

To determine the values of x for which a power series converges, we normally use the Ratio Test.

Example. For what values of x does the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ converge?

1. Let $a_n = \dfrac{(x-3)^n}{n}$

Apply the Ratio Test:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(x-3)^{n+1}}{n+1} \cdot \frac{n}{(x-3)^n} \right| = \lim_{n\to\infty} \frac{n}{n+1} \cdot |x-3| = |x-3|$$

By the Ratio Test, the series converges when $|x-3| < 1$ and diverges when $|x-3| > 1$.

2. Solving for x, $|x-3| < 1 \Rightarrow -1 < x-3 < 1 \Rightarrow 2 < x < 4$

3. Check the endpoints by hand (since the ratio test is inconclusive)

If $x = 2$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges by the A.S.T.

If $x = 4$: $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (harmonic series)

4. Final Answer: $2 \leq x < 4$

Example. For what values of x is the series $\sum_{n=0}^{\infty} n!x^n$ convergent?

Let $a_n = n!\,x^n$

Apply the Ratio Test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!\,x^{n+1}}{n!\,x^n}\right| = \lim_{n\to\infty} |(n+1)x|$$

$$= \lim_{n\to\infty} (n+1)\cdot|x|$$

$$= \infty \quad \text{if } x \neq 0$$

This series diverges for $x \neq 0$ and converges only when $x = 0$

Example. For what values of x does the series $\sum_{n=0}^{\infty} \frac{x^n}{(2n)!}$ converge?

Let $a_n = \dfrac{x^n}{(2n)!}$

Apply the Ratio Test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{(2n+2)!}\cdot\frac{(2n)!}{x^n}\right|$$

$$= \lim_{n\to\infty} \left|\frac{x}{(2n+2)(2n+1)}\right|$$

$$= \lim_{n\to\infty} \frac{|x|}{(2n+2)(2n+1)} = 0$$

By the Ratio Test, the series converges for all x.

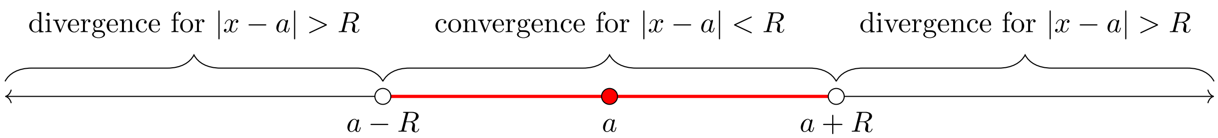
Interval of Convergence

Theorem. For a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, there are only three possibilities:

(i) The series converges only when $x = a$.

(ii) The series converges for all x.

(iii) There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.



Definition. The number R in case (iii) is called the **radius of convergence** of the power series. By convention:

- In case (i), the radius of convergence is $R = 0$.
- In case (ii), The radius of convergence is $R = \infty$.

Definition. The **interval of convergence** of a power series is the interval that consists of all values of x for which the series converges:

- In case (i), the interval is just a single point a.
- In case (ii), the interval is $(-\infty, \infty)$.
- In case (iii), the inequality $|x-a| < R$ can be rewritten as $a - R < x < a + R$.
  - Anything can happen at the endpoints $x = a + R$ or $x = a - R$.
  - The series could converge at one or both endpoints, or the series could diverge at one or both endpoints
  - In particular, there are four possibilities for the interval of convergence:

$$(a-R, a+R), \quad (a-R, a+R], \quad [a-R, a+R), \quad [a-R, a+R].$$

Example. Below is a table summarizing the convergence for the series we have seen so far.

Series Formula	Radius of Convergence	Interval of Convergence
$\sum_{n=0}^{\infty} x^n$	$R = 1$	$(-1, 1)$
$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$	$R = 1$	$[2, 4)$
$\sum_{n=0}^{\infty} n!x^n$	$R = 0$	$\{0\}$
$\sum_{n=0}^{\infty} \frac{x^n}{(2n)!}$	$R = \infty$	$(-\infty, \infty)$

Remark. In general, the Ratio Test should be used to determine the radius of convergence R. The Ratio Test always fails when x is an endpoint of the interval of convergence, so the endpoints must be checked with some other test.

Example. Find the radius of convergence and interval of convergence of the series:

$$\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}.$$

Let $a_n = \dfrac{(-3)^n x^n}{\sqrt{n+1}}$

Apply the Ratio Test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-3)^{n+1}x^{n+1}}{\sqrt{n+2}}\cdot\frac{\sqrt{n+1}}{(-3)^n x^n}\right| = \lim_{n\to\infty}\left|\frac{(-3)^{n+1}}{(-3)^n}\cdot\frac{x^{n+1}}{x^n}\cdot\frac{\sqrt{n+1}}{\sqrt{n+2}}\right|$$

$$= \lim_{n\to\infty}\left|-3x\cdot\sqrt{\frac{n+1}{n+2}}\right| = \lim_{n\to\infty} 3\cdot|x|\cdot\sqrt{\frac{n+1}{n+2}} = 3|x|$$

By the Ratio Test, the series converges for $3|x|<1$ and diverges for $3|x|>1$

Solve for x: $3|x|<1 \Rightarrow |x|<\frac{1}{3} \Rightarrow -\frac{1}{3}<x<\frac{1}{3}$

Check the endpoints:

When $x=-\frac{1}{3}$:

$$\sum_{n=0}^{\infty}\frac{(-3)^n\left(-\frac{1}{3}\right)^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty}\frac{1}{\sqrt{n+1}} \quad \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$$

which diverges (p-series with $p=\frac{1}{2}<1$)

When $x=\frac{1}{3}$:

$$\sum_{n=0}^{\infty}\frac{(-3)^n\left(\frac{1}{3}\right)^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{n+1}}$$

Converges by A.S.T.

Final Answer: the interval of convergence is $\left(-\frac{1}{3},\frac{1}{3}\right]$

the radius of convergence is $R=\frac{1}{3}$

Example. Find the radius of convergence and interval of convergence of the series:

$$\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}.$$