11.6 The Ratio Test

One way to determine how quickly the terms of a series are decreasing (or increasing) is to calculate the ratios of consecutive terms. For a geometric series $\sum ar^{n-1}$, we have $\left|\frac{a_{n+1}}{a_n}\right| = |r|$ for all n, and the series converges if $|r| < 1$. The Ratio Test tells us that for any series, if the ratios $\left|\frac{a_{n+1}}{a_n}\right|$ approach a number less than 1 as $n \to \infty$, then the series converges.

Theorem (Ratio Test)**.** Let $\sum a_n$ be a series with terms a_n. Define the limit

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|.$$

Then:

1. If $L < 1$, the series $\sum a_n$ is absolutely convergent (and therefore convergent).
2. If $L > 1$ or $L = \infty$, the series $\sum a_n$ is divergent.
3. If $L = 1$, the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of $\sum a_n$.

Proof.

- If $L < 1$, show the series converges absolutely.

- If $L > 1$ or $L = \infty$, show the series diverges.

- If $L = 1$, show the test is inconclusive.

$\square$

Example. Test the series $\sum_{n=1}^{\infty}(-1)^n \frac{n^3}{3^n}$ for absolute convergence.

Example. Test the convergence of the series $\sum_{n=1}^{\infty} \frac{n^n}{n!}$.

Remark. Although the Ratio Test works in the previous example, an easier method is to use the Test for Divergence. Since

$$a_n = \frac{n^n}{n!} = \frac{n \cdot n \cdot n \cdot \cdots \cdot n}{1 \cdot 2 \cdot 3 \cdot \cdots \cdot n} \geq n,$$

it follows that a_n does not approach 0 as $n \to \infty$. Therefore, the given series diverges.

Example. Use the ratio test to test the convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Example. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\arctan(n)}{2^n}$ is absolutely convergent, conditionally convergent, or divergent.