Ratio Test

1. $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{n^2}$

   Let $a_n = \frac{5^n}{n^2}$. Then

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\frac{5^{n+1}}{(n+1)^2}\cdot\frac{n^2}{5^n}\\ &= 5\cdot\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^2 = 5.\end{aligned}$$

   Since the limit is greater than 1, the series diverges.

2. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

   Let $a_n = \frac{n^2}{2^n}$. Then

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\frac{(n+1)^2}{2^{n+1}}\cdot\frac{2^n}{n^2}\\ &= \frac{1}{2}\cdot\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^2 = \frac{1}{2}.\end{aligned}$$

   The series converges absolutely.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^n}$

   Let $a_n = \frac{2^n}{n^n}$. Then

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\frac{2^{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{2^n}\\ &= 2\cdot\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n\cdot\frac{1}{n+1} = 0.\end{aligned}$$

   The series converges absolutely.

4. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$

   Let $a_n = \frac{2^n}{n!}$. Then

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\frac{2^{n+1}}{(n+1)!}\cdot\frac{n!}{2^n}\\ &= \lim_{n\to\infty}\frac{2}{n+1} = 0.\end{aligned}$$

   The series converges absolutely.

5. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{3^n}$

   Let $a_n = \frac{n!}{3^n}$. Then

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\frac{(n+1)!}{3^{n+1}}\cdot\frac{3^n}{n!}\\ &= \lim_{n\to\infty}\frac{n+1}{3} = \infty.\end{aligned}$$

   The series diverges.

6. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n \cdot n!}{n^n}$

Let $a_n = \frac{3^n n!}{n^n}$. Then

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \frac{3^{n+1}(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{3^n n!} \\ &= 3 \cdot \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = 3 \cdot \frac{1}{e}.\end{aligned}$$

Since $3/e > 1$, the series diverges.

7. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{(2n)!}$

Let $a_n = \frac{n!}{(2n)!}$. Then

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \frac{(n+1)!}{(2n+2)!} \cdot \frac{(2n)!}{n!} \\ &= \lim_{n\to\infty} \frac{(n+1)}{(2n+2)(2n+1)} = 0.\end{aligned}$$

The series converges absolutely.

8. $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{n! \cdot n^n}$

Let $a_n = \frac{(2n)!}{n! \cdot n^n}$. Then

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \frac{(2n+2)!}{(n+1)! \cdot (n+1)^{n+1}} \cdot \frac{n! \cdot n^n}{(2n)!} \\ &= \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{(n+1) \cdot (n+1)^{n+1}} \cdot n^n \\ &= 0.\end{aligned}$$

The series converges absolutely.

9. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \cdot \frac{n}{2^n}$

Let $a_n = \left|(-1)^n \cdot \frac{n}{2^n}\right| = \frac{n}{2^n}$. Then

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} \\ &= \frac{1}{2} \cdot \lim_{n\to\infty}\left(1 + \frac{1}{n}\right) = \frac{1}{2}.\end{aligned}$$

The series converges absolutely.

10. $\sum_{n=1}^{\infty}(-1)^n \cdot \frac{n^2}{n!}$

Let $a_n = \left|(-1)^n \cdot \frac{n^2}{n!}\right| = \frac{n^2}{n!}$. Then

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\frac{(n+1)^2}{(n+1)!}\cdot\frac{n!}{n^2}\\ &= \lim_{n\to\infty}\frac{(n+1)^2}{n+1}\cdot\frac{1}{n^2}\\ &= \lim_{n\to\infty}\frac{n+1}{n^2} = 0.\end{aligned}$$

The series converges absolutely.