11.6 The Ratio Test

One way to determine how quickly the terms of a series are decreasing (or increasing) is to calculate the ratios of consecutive terms. For a geometric series $\sum ar^{n-1}$, we have $\left|\frac{a_{n+1}}{a_n}\right| = |r|$ for all n, and the series converges if $|r| < 1$. The Ratio Test tells us that for any series, if the ratios $\left|\frac{a_{n+1}}{a_n}\right|$ approach a number less than 1 as $n \to \infty$, then the series converges.

Theorem (Ratio Test)**.** Let $\sum a_n$ be a series with terms a_n. Define the limit

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|.$$

Then:

1. If $L < 1$, the series $\sum a_n$ is absolutely convergent (and therefore convergent).
2. If $L > 1$ or $L = \infty$, the series $\sum a_n$ is divergent.
3. If $L = 1$, the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of $\sum a_n$.

Proof.

- If $L < 1$, show the series converges absolutely.

- If $L < 1$ choose a number r such that $L < r < 1$
- For sufficently large n, $\left|\frac{a_{n+1}}{a_n}\right| < r$
- This implies $|a_{n+1}| < |a_n| \cdot r$

$$|a_{n+2}| < |a_{n+1}| \cdot r < |a_n| \cdot r^2$$

$$\vdots$$

$$|a_{n+k}| < \cdots < |a_n| \cdot r^k \quad \text{for all } k \geq 1$$

- In particular, $\sum_{k=1}^{\infty} |a_{n+k}| \leq \sum_{k=1}^{\infty} |a_n| \cdot r^k$, so the tail of the series $\sum_{n=1}^{\infty} |a_n|$ is bounded by a geometric series that converges, since $r < 1$.

- Therefore, $\sum a_n$ converges absolutely.

- If $L > 1$ or $L = \infty$, show the series diverges.

- If $L > 1$ or $L = \infty$, then the ratio $\left|\frac{a_{n+1}}{a_n}\right|$ will eventually exceed 1.

- So there is some N such that for all $n \geq N$

$$\left|\frac{a_{n+1}}{a_n}\right| > 1 \quad \Rightarrow \quad |a_{n+1}| > |a_n| \quad \star$$

- Thus $\lim_{n\to\infty} a_n \neq 0$ and $\sum a_n$ diverges by the Test for Divergence.

- If $L = 1$, show the test is inconclusive.

The harmonic series $\sum \frac{1}{n}$ diverges but

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+1} = 1$$

The series $\sum \frac{1}{n^2}$ converges (p-series with $p = 2 > 1$)

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^2 = 1$$

$\square$

Example. Test the series $\sum_{n=1}^{\infty}(-1)^n \frac{n^3}{3^n}$ for absolute convergence.

We apply the Ratio Test by letting $a_n = (-1)^n \frac{n^3}{3^n}$

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} \cdot \frac{(n+1)^3}{3^{n+1}}}{(-1)^n \cdot \frac{n^3}{3^n}} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}}{(-1)^n} \cdot \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3} \right|$$

$$= \lim_{n\to\infty} \frac{(n+1)^3}{n^3} \cdot \frac{3^n}{3^{n+1}} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^3 \cdot \frac{1}{3} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^3 \cdot \frac{1}{3} = \frac{1}{3}$$

Since $\frac{1}{3} < 1$, this converges absolutely by the Ratio Test.

Example. Test the convergence of the series $\sum_{n=1}^{\infty} \frac{n^n}{n!}$.

Apply the Ratio Test. Since the terms $a_n = \frac{n^n}{n!}$ are positive, we don't need absolute values.

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+1)^{n+1}}{(n+1)!}}{\frac{n^n}{n!}} = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n}$$

$$= \lim_{n\to\infty} \frac{(n+1)^{n+1}}{n^n} \cdot \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{(n+1)(n+1)^n}{n^n} \cdot \frac{n!}{(n+1)\,n!}$$

e.g. $100! = 100 \cdot 99!$

$$= \lim_{n\to\infty} \frac{(n+1)^n}{n^n} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$$

indeterminate form 1^∞
L'Hôpital's Rule
let $L = (1+\frac{1}{n})^n$
$\ln(L) \to 1$, so $L \to e$

Since $e > 1$, the series diverges by the Ratio Test.

Remark. Although the Ratio Test works in the previous example, an easier method is to use the Test for Divergence. Since

$$a_n = \frac{n^n}{n!} = \frac{n \cdot n \cdot n \cdot \dots \cdot n}{1 \cdot 2 \cdot 3 \cdot \dots \cdot n} \geq n,$$

it follows that a_n does not approach 0 as $n \to \infty$. Therefore, the given series diverges.

Example. Use the ratio test to test the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. ← We did this above.

The series $\sum_{1}^{\infty} \frac{1}{n^2}$ converges (p-series with $p=2>1$)

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^2 = 1$$

Since $L=1$, the Ratio Test is inconclusive.

Example. Determine whether the series $\sum_{n=1}^{\infty} (-1)^n \frac{\arctan(n)}{2^n}$ is absolutely convergent, conditionally convergent, or divergent.

Apply the Ratio Test:

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}\arctan(n+1)}{2^{n+1}} \cdot \frac{2^n}{(-1)^n \arctan(n)} \right|$$

As $n\to\infty$, $\arctan(n) \to \frac{\pi}{2}$

$$= \lim_{n\to\infty} \left| \frac{\arctan(n+1)}{\arctan(n)} \cdot \frac{1}{2} \right| = 1 \cdot \frac{1}{2} = \frac{1}{2}$$

Since $L<1$, the series converges absolutely.