11.5 Alternating Series and Absolute Convergence

The convergence tests that we have looked at so far apply only to series with positive terms. We now begin examining series whose terms are not necessarily positive. Of particular importance are alternating series, whose terms alternate in sign.

Alternating Series

Definition. An **alternating series** is a series whose terms are alternately positive and negative. Formally, it is a series of the form:

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n \quad \text{or} \quad \sum_{n=1}^{\infty}(-1)^{n}b_n,$$

where $b_n > 0$ for all n.

Example. Here are two examples of alternating series:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$$

$$-\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} - \frac{5}{6} + \frac{6}{7} - \cdots = \sum_{n=1}^{\infty}(-1)^{n}\frac{n}{n+1}$$

Theorem (Alternating Series Test)**.** If the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots \quad (b_n > 0)$$

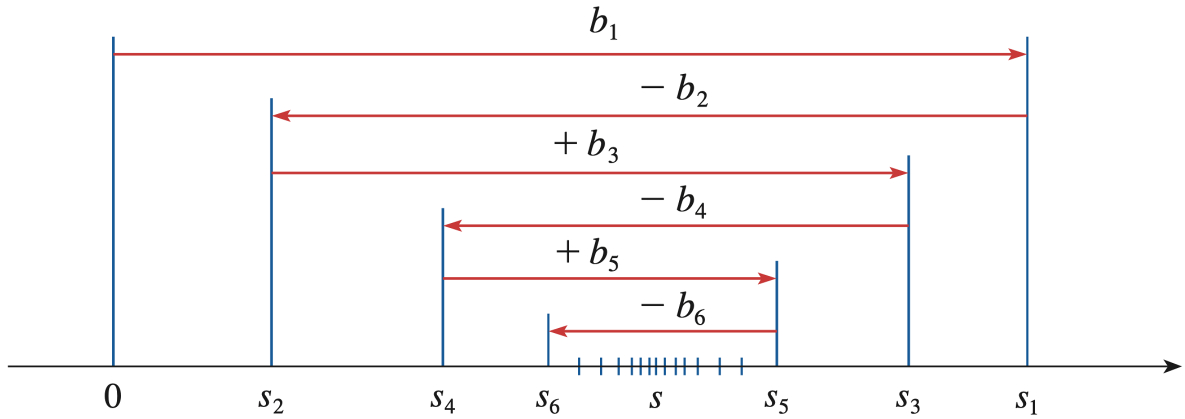
satisfies the conditions:

(i) $b_{n+1} \leq b_n$ for all n,

(ii) $\lim_{n\to\infty} b_n = 0$,

then the series is convergent. In other words, if the terms of an alternating series decrease toward 0 in absolute value, then the series converges.

Visual Proof of the Alternating Series Test



- Start by plotting the first term $s_1 = b_1$ on a number line.
- To find s_2, subtract b_2. This places s_2 to the left of s_1.
- Add b_3 to find s_3, which moves s_3 to the right of s_2.
- Subtract b_4 to locate s_4, which places s_4 to the left of s_3.
- Continue this process, alternating addition and subtraction of the terms b_n.
- Since $b_n \to 0$, the successive steps become smaller and smaller. The partial sums oscillate back and forth.
- Observe:
  - The even partial sums $s_2, s_4, s_6, \ldots$ form an increasing sequence.
  - The odd partial sums $s_1, s_3, s_5, \ldots$ form a decreasing sequence.
- Both the even and odd partial sums converge to the same limit s, which is the sum of the series.

Proof of the Alternating Series Test

- Why are the even partial sums $s_2, s_4, s_6, \ldots$ increasing and bounded above?

- What can we conclude about the limit of the even partial sums $s_2, s_4, s_6, \ldots$?

- Why do the odd partial sums $s_1, s_3, s_5, \ldots$ converge to the same limit s?

- What does this imply about the convergence of the series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$?

Example. Determine whether the alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

converges or diverges.

Example. Determine whether the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n - 1}$$

converges or diverges.

Example. Test the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{n^3+1}$ for convergence or divergence.

Estimating Sums of Alternating Series

A partial sum s_n of any convergent series can be used as an approximation to the total sum s. However, this is only useful if we can estimate the accuracy of the approximation. The error involved in approximating s by s_n is called the remainder:

$$R_n = s - s_n.$$

The following theorem provides a bound for the size of this error for series that satisfy the conditions of the Alternating Series Test. Specifically, the error is smaller than b_{n+1}, the absolute value of the first neglected term.

Theorem. If $s = \sum_{n=1}^{\infty} (-1)^{n-1} b_n$, where $b_n > 0$, is the sum of an alternating series that satisfies:

(i) $b_{n+1} \leq b_n$

(ii) $\lim_{n\to\infty} b_n = 0$,

then the remainder $R_n = s - s_n$ satisfies:

$$|R_n| = |s - s_n| \leq b_{n+1},$$

where b_{n+1} is the absolute value of the first neglected term.

Proof.

□

Example. Find the sum of the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ correct to three decimal places.

Remark. The rule that the error $|s - s_n|$ is smaller than the first neglected term b_{n+1} is valid only for alternating series that satisfy the conditions of the Alternating Series Estimation Theorem. This rule does not apply to other types of series.

Absolute Convergence and Conditional Convergence

Definition. Given any series $\sum a_n$, we can consider the corresponding series:

$$\sum |a_n| = |a_1| + |a_2| + |a_3| + \cdots ,$$

whose terms are the absolute values of the terms of the original series.

A series $\sum a_n$ is called **absolutely convergent** if the series of absolute values $\sum |a_n|$ is convergent.

Notice that if $\sum a_n$ is a series with positive terms, then $|a_n| = a_n$, and so absolute convergence is the same as convergence in this case.

Example. Determine whether the alternating series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

is absolutely convergent.

Definition. A series $\sum a_n$ is called **conditionally convergent** if it is convergent but not absolutely convergent; that is, if $\sum a_n$ converges but $\sum |a_n|$ diverges.

Example. Show that the that the alternating harmonic series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

is conditionally convergent.

Remark. This example shows that it is possible for a series to be convergent but not absolutely convergent. The following theorem states that absolute convergence implies convergence.

Theorem. If a series $\sum a_n$ is absolutely convergent, then it is convergent.

$\square$

Example. Determine whether the series

$$\sum_{n=1}^{\infty} \frac{\cos n}{n^2} = \frac{\cos 1}{1^2} + \frac{\cos 2}{2^2} + \frac{\cos 3}{3^2} + \cdots$$

is convergent or divergent.

Example. Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$ is absolutely convergent, conditionally convergent, or divergent.

Example. Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}$ is absolutely convergent, conditionally convergent, or divergent.

Example. Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^n n}{2n+1}$ is absolutely convergent, conditionally convergent, or divergent.