

11.5 Estimating Sums of Alternating Series

In general, we do not know the exact sum of an infinite alternating series. So a natural question is: *if we cannot find the sum exactly, how can we estimate it?* Using a partial sum S_n is a reasonable place to start, and The Alternating Series Estimation Theorem tells us that the difference between this partial sum and the true sum S is no bigger than the first term we leave out.

Theorem. Let $S = \sum_{n=1}^{\infty} (-1)^{n-1} b_n$ be a convergent alternating series, and let S_n denote its n th partial sum. Then the remainder

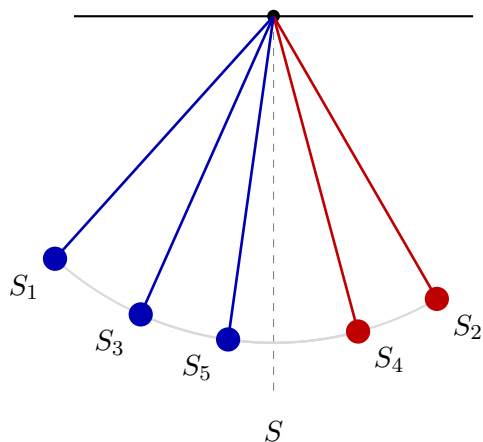
$$R_n = S - S_n$$

satisfies

$$|R_n| = |S - S_n| \leq b_{n+1},$$

where b_{n+1} is the absolute value of the first neglected term.

Proof. One helpful way to visualize a convergent alternating series is to think of its partial sums as the turning points of a pendulum. In this picture, the equilibrium position represents the true sum S , and the length of each swing represents the size of the next term b_{n+1} . Each time the pendulum swings, it passes through its equilibrium position, then turns around on the other side, with each swing a little smaller than the one before.



□

Example. Suppose we use S_4 to approximate the sum of the alternating series

$$S = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}.$$

Use the Alternating Series Estimation Theorem to find an upper bound for the difference between S and S_4 .

Example. How many terms of the alternating series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

are needed to guarantee that the partial sum is within 0.001 of the true sum?