**Remainder Estimates (Solutions)**

1. Approximate $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ using the first 4 terms.

$$\begin{aligned} S_4 &= 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} \\ &= \frac{115}{144} \\ &\approx 0.7986. \end{aligned}$$

By the Alternating Series Estimation Theorem,

$$|R_4| \leq b_5 = \frac{1}{5^2} = \frac{1}{25} = 0.04.$$

2. Approximate $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n}}{n^3+1}$ using the first 3 terms.

$$\begin{aligned} S_3 &= -\frac{1}{2} + \frac{1}{9} - \frac{1}{28} \\ &= -\frac{107}{252} \\ &\approx -0.4246. \end{aligned}$$

Thus,

$$|R_3| \leq b_4 = \frac{1}{4^3+1} = \frac{1}{65} \approx 0.0154.$$

3. How many terms are needed to approximate $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^5}$ to within 0.0001?

We want

$$|R_n| \leq b_{n+1} = \frac{1}{(n+1)^5} < 0.0001.$$

So

$$\begin{aligned} \frac{1}{(n+1)^5} < 0.0001 &\iff (n+1)^5 > 10000 \\ &\iff n+1 > \sqrt[5]{10000} \approx 6.31. \end{aligned}$$

Therefore, the smallest such $n$ is $n = 6$, so $\boxed{\text{6 terms}}$ are needed.

4. Use the first 5 terms of $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\ln(n+1)}{n^2}$ to approximate the sum.

$$\begin{aligned} S_5 &= -\frac{\ln 2}{1^2} + \frac{\ln 3}{2^2} - \frac{\ln 4}{3^2} + \frac{\ln 5}{4^2} - \frac{\ln 6}{5^2} \\ &\approx -0.6931 + 0.2747 - 0.1540 + 0.1006 - 0.0717 \\ &\approx -0.5436. \end{aligned}$$

Thus,

$$|R_5| \leq b_6 = \frac{\ln 7}{6^2} \approx 0.0541.$$

5. Determine the minimum number of terms needed to estimate $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$ with error less than 0.01.

We need

$$|R_n| \leq b_{n+1} = \frac{1}{(n+1)\ln(n+1)} < 0.01.$$

Checking nearby values,

$$\frac{1}{29 \ln 29} > 0.01, \qquad \frac{1}{30 \ln 30} < 0.01.$$

So the smallest $n$ is $n = 29$.

Since the series starts at $n = 2$, this means using terms $n = 2$ through $n = 29$, i.e. $\boxed{\text{28 terms}}$.

6. Approximate $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ using the first 5 terms.

$$\begin{aligned} S_5 &= \sum_{n=1}^{5} \frac{1}{n^2+1} \\ &= \frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \frac{1}{17} + \frac{1}{26} \\ &\approx 0.8973. \end{aligned}$$

Using the Integral Test Remainder Estimate,

$$\begin{aligned} R_5 &\leq \int_5^{\infty} \frac{1}{x^2+1}\, dx \\ &= \tan^{-1}(x)\big|_5^{\infty} \\ &= \frac{\pi}{2} - \tan^{-1}(5) \\ &\approx 0.1974. \end{aligned}$$

So the error is less than 0.1974.

7. Estimate the remainder when approximating

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$$

by its partial sum through $n = 10$.

Let $f(x) = \frac{1}{x(\ln x)^2}$. Then

$$\int_{11}^{\infty} f(x)\, dx \leq R_{10} \leq \int_{10}^{\infty} f(x)\, dx.$$

Using $u = \ln x$, $du = \frac{1}{x}\, dx$,

$$\begin{aligned} \int \frac{1}{x(\ln x)^2}\, dx &= \int u^{-2}\, du \\ &= -\frac{1}{u} + C \\ &= -\frac{1}{\ln x} + C. \end{aligned}$$

Thus,

$$\begin{aligned} \int_{10}^{\infty} \frac{1}{x(\ln x)^2}\, dx &= \frac{1}{\ln 10}, \\ \int_{11}^{\infty} \frac{1}{x(\ln x)^2}\, dx &= \frac{1}{\ln 11}. \end{aligned}$$

Therefore,

$$\boxed{\frac{1}{\ln 11} \leq R_{10} \leq \frac{1}{\ln 10}}.$$

8. Estimate $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ using the first 8 terms.

$$\begin{aligned}
S_8 &= \sum_{n=1}^{8} \frac{1}{n^{3/2}} \\
&\approx 1 + 0.3536 + 0.1925 + 0.1250 + 0.0894 + 0.0680 + 0.0540 + 0.0442 \\
&\approx 1.9267.
\end{aligned}$$

Also,

$$\begin{aligned}
R_8 &\leq \int_8^{\infty} \frac{1}{x^{3/2}}\, dx \\
&= -2x^{-1/2}\Big|_8^{\infty} \\
&= \frac{2}{\sqrt{8}} \\
&\approx 0.7071.
\end{aligned}$$

9. Find the smallest $n$ such that $\displaystyle\sum_{k=2}^{n} \frac{1}{k(\ln k)^2}$ approximates the full sum within 0.05.

We want

$$R_n \leq \int_n^{\infty} \frac{1}{x(\ln x)^2}\, dx < 0.05.$$

Using $u = \ln x$, $du = \dfrac{1}{x}\, dx$,

$$\begin{aligned}
\int_n^{\infty} \frac{1}{x(\ln x)^2}\, dx &= \int_{\ln n}^{\infty} \frac{1}{u^2}\, du \\
&= \frac{1}{\ln n}.
\end{aligned}$$

So

$$\begin{aligned}
\frac{1}{\ln n} < 0.05 &\iff \ln n > 20 \\
&\iff n > e^{20}.
\end{aligned}$$

Therefore, the smallest integer $n$ is

$$\boxed{n = 485{,}165{,}196}.$$

10. Approximate $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n^3}$ using the first 6 terms.

$$\begin{aligned}
S_6 &= \sum_{n=2}^{7} \frac{\ln n}{n^3} \\
&\approx 0.0866 + 0.0407 + 0.0217 + 0.0129 + 0.0083 + 0.0057 \\
&\approx 0.1758.
\end{aligned}$$

For the error,

$$R_6 \leq \int_7^{\infty} \frac{\ln x}{x^3}\, dx.$$

Using integration by parts with

$$u = \ln x, \quad dv = x^{-3}\,dx, \qquad du = \frac{1}{x}\,dx, \quad v = -\frac{1}{2x^2},$$

we get

$$\begin{aligned}\int \frac{\ln x}{x^3}\,dx &= -\frac{\ln x}{2x^2} + \frac{1}{2}\int x^{-3}\,dx \\ &= -\frac{\ln x}{2x^2} - \frac{1}{4x^2}.\end{aligned}$$

Therefore,

$$\begin{aligned}R_6 &\leq -\frac{\ln x}{2x^2} - \frac{1}{4x^2}\bigg|_7^\infty \\ &= \frac{\ln 7}{98} + \frac{1}{196} \\ &\approx 0.0250.\end{aligned}$$

So the error is less than 0.0250.