## 11.5 Estimating Sums of Alternating Series

In general, we do not know the exact sum of an infinite alternating series. So a natural question is: *if we cannot find the sum exactly, how can we estimate it?* Using a partial sum $S_n$ is a reasonable place to start, and The Alternating Series Estimation Theorem tells us that the difference between this partial sum and the true sum $S$ is no bigger than the first term we leave out.

**Theorem.** Let $S = \sum_{n=1}^{\infty}(-1)^{n-1}b_n$ be a convergent alternating series, and let $S_n$ denote its $n$th partial sum. Then the remainder

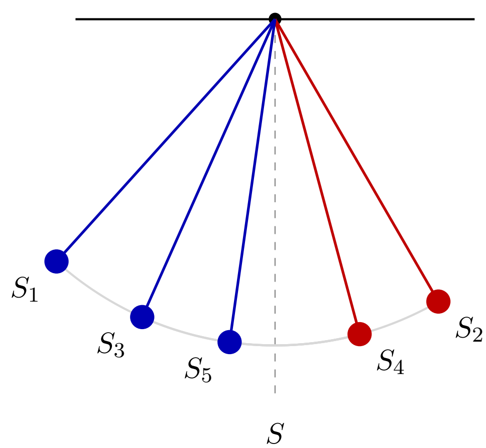
$$R_n = S - S_n$$

satisfies

$$|R_n| = |S - S_n| \leq b_{n+1},$$

where $b_{n+1}$ is the absolute value of the first neglected term.

*Proof.* One helpful way to visualize a convergent alternating series is to think of its partial sums as the turning points of a pendulum. In this picture, the equilibrium position represents the true sum $S$, and the length of each swing represents the size of the next term $b_{n+1}$. Each time the pendulum swings, it passes through its equilibrium position, then turns around on the other side, with each swing a little smaller than the one before.




When we add the term $b_{n+1}$ to $S_n$ we always pass through the equilibrium position S. Hence

$$\underbrace{|S - S_n|}_{\text{distance to equilibrium}} \leq \underbrace{b_{n+1}}_{\text{length of the next swing}}$$

□

**Example.** Suppose we use $S_4$ to approximate the sum of the alternating series

$$S = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}.$$

Use the Alternating Series Estimation Theorem to find an upper bound for the difference between $S$ and $S_4$.

$$|S - S_4| \leq b_5 = \frac{1}{5}$$

In other words, $S_4 = \underbrace{-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4}}_{-0.5833}$ is within 0.2 of $\underbrace{\sum_{n=1}^{\infty} \frac{(-1)^n}{n}}_{-0.6931}$

**Example.** How many terms of the alternating series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

are needed to guarantee that the partial sum is within 0.001 of the true sum?

We need to find $N$ so that $b_{N+1} \leq 0.001$

$$\Rightarrow \frac{1}{N+1} \leq 0.001$$

$$\Rightarrow 1000 \leq N+1$$

$$\Rightarrow 999 \leq N$$

Conclude: We need 999 terms and $S_{999}$ is within 0.001 of $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$