11.5 Absolute Convergence

Definition. Let $\sum a_n$ be a series. The series

$$\sum |a_n| = |a_1| + |a_2| + |a_3| + \cdots$$

obtained by taking the absolute value of each term is called the **series of absolute values**. We say that the series $\sum a_n$ is **absolutely convergent** if the series

$$\sum |a_n|$$

converges. If $\sum a_n$ is a series with positive terms, then $|a_n| = a_n$ for all n, so absolute convergence is equivalent to convergence.

Example. Determine whether the alternating series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

is absolutely convergent.

Definition. A series $\sum a_n$ is called **conditionally convergent** if it converges, but does not converge absolutely. In other words, $\sum a_n$ converges while $\sum |a_n|$ diverges.

Example. Show that the that the alternating harmonic series

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

is conditionally convergent.

Theorem. If a series $\sum a_n$ is absolutely convergent, then it is convergent.

Example. Determine whether the series

$$\sum_{n=1}^{\infty} \frac{\cos n}{n^2} = \frac{\cos 1}{1^2} + \frac{\cos 2}{2^2} + \frac{\cos 3}{3^2} + \cdots$$

is convergent or divergent.

Example. Determine whether the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$$

is absolutely convergent, conditionally convergent, or divergent.

Example. Determine whether the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}$$

is absolutely convergent, conditionally convergent, or divergent.

Example. Determine whether the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{2n+1}$$

is absolutely convergent, conditionally convergent, or divergent.