

## Absolute Convergence

1. 
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

Let

$$b_n = \frac{1}{n}.$$

Then

$$b_n > 0, \quad \lim_{n \rightarrow \infty} b_n = 0,$$

and  $b_n$  is decreasing since  $\frac{1}{n+1} < \frac{1}{n}$  for all  $n \geq 1$ . Therefore,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

converges by the Alternating Series Test.

To test for absolute convergence, consider

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}.$$

This is the harmonic series, which diverges.

**Conclusion:** The series **converges conditionally**.

2. 
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

Consider the series of absolute values:

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

This is a  $p$ -series with  $p = 2$ , and since  $p > 1$ , it converges.

Therefore,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

converges absolutely.

**Conclusion:** The series **converges absolutely**.

3. 
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

Let

$$b_n = \frac{1}{\sqrt{n}}.$$

Then

$$b_n > 0, \quad \lim_{n \rightarrow \infty} b_n = 0,$$

and  $b_n$  is decreasing since  $\frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}$  for all  $n \geq 1$ . Therefore,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

converges by the Alternating Series Test.

To test for absolute convergence, consider

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}.$$

This is a  $p$ -series with  $p = \frac{1}{2}$ , and since  $p \leq 1$ , it diverges.

**Conclusion:** The series **converges conditionally**.

4. 
$$\sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{n}$$

Let

$$b_n = \frac{\ln n}{n}.$$

For  $n \geq 2$ , we have  $b_n \geq 0$ , and

$$\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{\ln n}{n} = 0.$$

Also,

$$f(x) = \frac{\ln x}{x} \quad \Rightarrow \quad f'(x) = \frac{1 - \ln x}{x^2}.$$

Thus  $f'(x) < 0$  when  $x > e$ , so  $b_n$  is decreasing for  $n \geq 3$ . Therefore,

$$\sum_{n=3}^{\infty} \frac{(-1)^n \ln n}{n}$$

converges by the Alternating Series Test, and hence

$$\sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{n}$$

also converges.

To test for absolute convergence, consider

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n \ln n}{n} \right| = \sum_{n=1}^{\infty} \frac{\ln n}{n}.$$

Using the Integral Test with  $f(x) = \frac{\ln x}{x}$ ,

$$\begin{aligned} \int_1^{\infty} \frac{\ln x}{x} dx &= \lim_{b \rightarrow \infty} \int_1^b \frac{\ln x}{x} dx \\ &= \lim_{b \rightarrow \infty} \left[ \frac{(\ln x)^2}{2} \right]_1^b \\ &= \infty. \end{aligned}$$

So

$$\sum_{n=1}^{\infty} \frac{\ln n}{n}$$

diverges.

**Conclusion:** The series **converges conditionally**.

5. 
$$\sum_{n=1}^{\infty} \frac{\sin(n^2 + 1)}{n^3}$$

Consider the series of absolute values:

$$\sum_{n=1}^{\infty} \left| \frac{\sin(n^2 + 1)}{n^3} \right|.$$

Since

$$|\sin(n^2 + 1)| \leq 1$$

for all  $n$ , we have

$$\left| \frac{\sin(n^2 + 1)}{n^3} \right| \leq \frac{1}{n^3}.$$

Because

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

is a  $p$ -series with  $p = 3 > 1$ , it converges. Therefore, by the Comparison Test,

$$\sum_{n=1}^{\infty} \left| \frac{\sin(n^2 + 1)}{n^3} \right|$$

converges.

**Conclusion:** The series **converges absolutely**.

6. 
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)\ln(n+1)}$$

Let

$$b_n = \frac{1}{(n+1)\ln(n+1)}.$$

Then  $b_n > 0$ ,  $b_n \rightarrow 0$ , and  $b_n$  is decreasing for  $n \geq 1$ . Therefore, by the Alternating Series Test,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)\ln(n+1)}$$

converges.

To test for absolute convergence, consider

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{(n+1)\ln(n+1)} \right| = \sum_{n=1}^{\infty} \frac{1}{(n+1)\ln(n+1)}.$$

Using the Integral Test with  $f(x) = \frac{1}{(x+1)\ln(x+1)}$ ,

$$\begin{aligned} \int_1^{\infty} \frac{1}{(x+1)\ln(x+1)} dx &= \int_{\ln 2}^{\infty} \frac{1}{u} du \\ &= \infty. \end{aligned}$$

So

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)\ln(n+1)}$$

diverges.

**Conclusion:** The series converges conditionally.

$$7. \sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$$

Consider the series of absolute values:

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n!} \right| = \sum_{n=1}^{\infty} \frac{1}{n!}.$$

Apply the Ratio Test:

$$\begin{aligned} L &= \lim_{n \rightarrow \infty} \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} \\ &= \lim_{n \rightarrow \infty} \frac{n!}{(n+1)!} \\ &= \lim_{n \rightarrow \infty} \frac{1}{n+1} \\ &= 0. \end{aligned}$$

Since  $L < 1$ , the series

$$\sum_{n=1}^{\infty} \frac{1}{n!}$$

converges. Therefore,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$$

converges absolutely.

**Conclusion:** The series **converges absolutely**.

$$8. \sum_{n=1}^{\infty} \frac{\cos(n)}{n^{3/2}}$$

Consider the series of absolute values:

$$\sum_{n=1}^{\infty} \left| \frac{\cos(n)}{n^{3/2}} \right|.$$

Since

$$|\cos(n)| \leq 1$$

for all  $n$ , we have

$$\left| \frac{\cos(n)}{n^{3/2}} \right| \leq \frac{1}{n^{3/2}}.$$

Because

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

is a  $p$ -series with  $p = \frac{3}{2} > 1$ , it converges. Therefore, by the Comparison Test,

$$\sum_{n=1}^{\infty} \left| \frac{\cos(n)}{n^{3/2}} \right|$$

converges.

**Conclusion:** The series **converges absolutely**.

9.  $\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)^2}$

Consider the series of absolute values:

$$\sum_{n=2}^{\infty} \left| \frac{(-1)^n}{n(\ln n)^2} \right| = \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}.$$

Use the Integral Test with

$$f(x) = \frac{1}{x(\ln x)^2}.$$

For  $x \geq 2$ ,  $f(x)$  is positive and continuous. Also,

$$\begin{aligned} f'(x) &= -\frac{(\ln x)^2 + 2 \ln x}{x^2(\ln x)^4} \\ &= -\frac{\ln x + 2}{x^2(\ln x)^3}. \end{aligned}$$

Since  $x \geq 2$  implies  $\ln x > 0$ , we have  $f'(x) < 0$ , so  $f(x)$  is decreasing on  $[2, \infty)$ .

Now compute:

$$\begin{aligned} \int_2^{\infty} \frac{1}{x(\ln x)^2} dx &= \int_{\ln 2}^{\infty} \frac{1}{u^2} du \\ &= \left[ -\frac{1}{u} \right]_{\ln 2}^{\infty} \\ &= \frac{1}{\ln 2}. \end{aligned}$$

Since this improper integral converges, the series

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$$

converges by the Integral Test. Therefore,

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)^2}$$

converges absolutely.

**Conclusion:** The series **converges absolutely**.

10.  $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$

Consider the series of absolute values:

$$\sum_{n=1}^{\infty} \left| \frac{\sin(n)}{n^2} \right|.$$

Since

$$|\sin(n)| \leq 1$$

for all  $n$ , we have

$$\left| \frac{\sin(n)}{n^2} \right| \leq \frac{1}{n^2}.$$

Because

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

is a  $p$ -series with  $p = 2 > 1$ , it converges. Therefore, by the Comparison Test,

$$\sum_{n=1}^{\infty} \left| \frac{\sin(n)}{n^2} \right|$$

converges.

**Conclusion:** The series **converges absolutely**.