11.5 Absolute Convergence

Definition. Let $\sum a_n$ be a series. The series

$$\sum |a_n| = |a_1| + |a_2| + |a_3| + \cdots$$

obtained by taking the absolute value of each term is called the **series of absolute values**. We say that the series $\sum a_n$ is **absolutely convergent** if the series

$$\sum |a_n|$$

converges. If $\sum a_n$ is a series with positive terms, then $|a_n| = a_n$ for all n, so absolute convergence is equivalent to convergence.

Example. Determine whether the alternating series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

is absolutely convergent.

Consider the series of absolute values:

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

This is a p-series with $p = 2 > 1$, which converges.

Therefore, $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ is absolutely convergent.

Definition. A series $\sum a_n$ is called **conditionally convergent** if it converges, but does not converge absolutely. In other words, $\sum a_n$ converges while $\sum |a_n|$ diverges.

Example. Show that the that the alternating harmonic series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

is conditionally convergent.

- Let $a_n = (-1)^{n-1} \cdot \frac{1}{n}$
- We showed $\sum_{n=1}^{\infty} a_n$ converges by the A.S.T.
- The series $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n}$ is the harmonic series, which diverges.
- Since $\sum_{n=1}^{\infty} a_n$ converges, but $\sum_{n=1}^{\infty} |a_n|$ diverges, the series $\sum_{n=1}^{\infty} a_n$

  is conditionally convergent.

Theorem. If a series $\sum a_n$ is absolutely convergent, then it is convergent.

Example. Determine whether the series

$$\sum_{n=1}^{\infty} \frac{\cos n}{n^2} = \frac{\cos 1}{1^2} + \frac{\cos 2}{2^2} + \frac{\cos 3}{3^2} + \cdots$$

is convergent or divergent.

Intro: We will test for absolute convergence. Consider the series of absolute values:

$$\sum_{n=1}^{\infty} \left| \frac{\cos n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{|\cos n|}{n^2}$$

Let $a_n = \frac{|\cos n|}{n^2}$ and $b_n = \frac{1}{n^2}$. Both are positive, so D.C.T. applies

Apply D.C.T: We have $\frac{|\cos n|}{n^2} \leq \frac{1}{n^2}$ for $n \geq 1$ since $|\cos n| \leq 1$.

Also, the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series ($p = 2 > 1$).

Conclusion: By the D.C.T., $\sum_{n=1}^{\infty} \frac{|\cos n|}{n^2}$ converges. Therefore, $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ is absolutely convergent, hence convergent.

Example. Determine whether the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$$

is absolutely convergent, conditionally convergent, or divergent.

$|(-1)^n| = 1$

Intro: Test for absolute convergence:

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^3} \right| = \sum_{n=1}^{\infty} \frac{1}{n^3}$$

Apply p-series Test:

$\sum_{n=1}^{\infty} |a_n|$ is a p-series with $p = 3 > 1$, which converges.

Conclusion:

Since $\sum_{n=1}^{\infty} |a_n|$ converges, $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

Example. Determine whether the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}$$

is absolutely convergent, conditionally convergent, or divergent.

① Intro: Test for absolute convergence:

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \left| \frac{(-1)^n}{\sqrt[3]{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$$

Apply p-series Test: $\sum_{n=1}^{\infty} |a_n|$ is a p-series with $p = \frac{1}{3} \leq 1$, which diverges.

Conclude: $\sum_{n=1}^{\infty} a_n$ does not converge absolutely.

② Intro: $\sum_{n=1}^{\infty} a_n$ is an alternating series, so the A.S.T. applies.

Apply A.S.T.:

① $b_{n+1} < b_n$ because $\frac{1}{\sqrt[3]{n+1}} < \frac{1}{\sqrt[3]{n}}$ ✓

② $\lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{1}{\sqrt[3]{n}} = 0$ ✓

Conclusion: The series $\sum_{n=1}^{\infty} a_n$ converges by the A.S.T.

Since $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges, $\sum_{n=1}^{\infty} a_n$ is conditionally convergent.

Example. Determine whether the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{2n+1}$$

is absolutely convergent, conditionally convergent, or divergent.

Intro: We will apply the Test for Divergence to show this diverges.

Apply the Test:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^n \cdot \frac{n}{2n+1}$$

As $n\to\infty$, the magnitude of a_n approaches $\lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2}$.

Conclusion: Since $\lim_{n\to\infty} a_n \neq 0$ (the terms oscillate between approx. $\pm\frac{1}{2}$) the series diverges by the Test for Divergence.