11.5 Alternating Series

The convergence tests that we have looked at so far apply only to series with positive terms. We now begin examining series whose terms are not necessarily positive. Of particular importance are alternating series, whose terms alternate in sign.

Definition. An **alternating series** is a series whose terms are alternately positive and negative. Formally, it is a series of the form:

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n \quad \text{or} \quad \sum_{n=1}^{\infty}(-1)^n b_n,$$

where $b_n > 0$ for all n.

Example. What are some examples of alternating series?

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots$$

$$\sum_{n=1}^{\infty} \frac{\sin(\frac{\pi}{2}n)}{n} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

Theorem (Alternating Series Test)**.** If the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots \quad (b_n > 0)$$

satisfies the conditions:

(i) $b_{n+1} \leq b_n$ for all n,

(ii) $\lim_{n\to\infty} b_n = 0$,

then the series is convergent. In other words, if the terms of an alternating series decrease toward 0 in absolute value, then the series converges.

Example. Determine whether the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \rightarrow \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1}{n}$$

$$b_n = \frac{1}{n}$$

converges or diverges.

Intro: This is an alternating series, so the A.S.T. applies.

Apply A.S.T.:

1. $b_{n+1} < b_n$ because $\frac{1}{n+1} < \frac{1}{n}$ ✓

2. $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0$ ✓

Conclusion: The series converges by the Alternating Series Test.

Example. Determine whether the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1}$$

converges or diverges.

The alternating series test will fail since $\lim_{n\to\infty} b_n \neq 0$:

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{3n}{4n-1} \cdot \frac{1/n}{1/n} = \lim_{n\to\infty} \frac{3}{4-1/n} = \frac{3}{4-0} = \frac{3}{4} \quad \times$$

Instead, consider the terms in the original series:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^n \cdot \frac{3n}{4n-1}$$

This limit is D.N.E. since the terms eventually oscillate between approx. $\pm \frac{3}{4}$. The series diverges by the Test for Divergence.

Example. Test the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{n^3+1}$ for convergence or divergence.

Intro: This is an alternating series, so the A.S.T. applies.

Apply A.S.T.:

1. $b_{n+1} < b_n$: Consider $f(x) = \frac{x^2}{x^3+1}$.

$$f'(x) = \frac{(x^3+1)\cdot 2x - x^2(3x^2)}{(x^3+1)^2} = \frac{2x^4+2x-3x^4}{(x^3+1)^2} = \frac{2x-x^4}{(x^3+1)^2} = \frac{x(2-x^3)}{(x^3+1)^2}$$

$f'(x) < 0$ when $2-x^3 < 0$, i.e. when $x > \sqrt[3]{2}$. Hence $b_{n+1} < b_n$ for $n \geq 2$.

2. $\lim_{n\to\infty} b_n = 0$:

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n^2}{n^3+1}\cdot\frac{1/n^3}{1/n^3} = \lim_{n\to\infty}\frac{1/n}{1+1/n^3} = \frac{0}{1+0} = 0$$

Conclusion: The series converges by the A.S.T.