11.5 Alternating Series and Absolute Convergence

The convergence tests that we have looked at so far apply only to series with positive terms. We now begin examining series whose terms are not necessarily positive. Of particular importance are alternating series, whose terms alternate in sign.

Alternating Series

Definition. An **alternating series** is a series whose terms are alternately positive and negative. Formally, it is a series of the form:

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n \quad \text{or} \quad \sum_{n=1}^{\infty}(-1)^{n}b_n,$$

where $b_n > 0$ for all n.

Example. Here are two examples of alternating series:

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$$
$$-\frac{1}{2}+\frac{2}{3}-\frac{3}{4}+\frac{4}{5}-\frac{5}{6}+\frac{6}{7}-\cdots=\sum_{n=1}^{\infty}(-1)^{n}\frac{n}{n+1}$$

Theorem (Alternating Series Test)**.** If the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1-b_2+b_3-b_4+b_5-b_6+\cdots \quad (b_n>0)$$

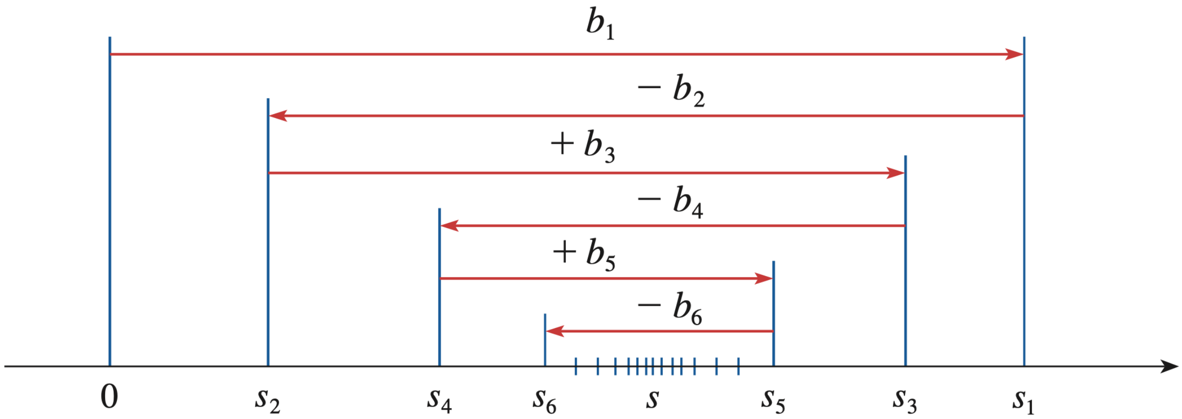
satisfies the conditions:

(i) $b_{n+1} \le b_n$ for all n,

(ii) $\lim_{n\to\infty} b_n = 0$,

then the series is convergent. In other words, if the terms of an alternating series decrease toward 0 in absolute value, then the series converges.

Visual Proof of the Alternating Series Test



- Start by plotting the first term $s_1 = b_1$ on a number line.
- To find s_2, subtract b_2. This places s_2 to the left of s_1.
- Add b_3 to find s_3, which moves s_3 to the right of s_2.
- Subtract b_4 to locate s_4, which places s_4 to the left of s_3.
- Continue this process, alternating addition and subtraction of the terms b_n.
- Since $b_n \to 0$, the successive steps become smaller and smaller. The partial sums oscillate back and forth.
- Observe:
  - The even partial sums $s_2, s_4, s_6, \ldots$ form an increasing sequence.
  - The odd partial sums $s_1, s_3, s_5, \ldots$ form a decreasing sequence.
- Both the even and odd partial sums converge to the same limit s, which is the sum of the series.

Proof of the Alternating Series Test

- Why are the even partial sums $s_2, s_4, s_6, \ldots$ increasing and bounded above?

$$s_2 = b_1 - b_2 \geq 0 \quad (\text{since } b_2 \leq b_1)$$

$$s_4 = s_2 + (b_3 - b_4) \geq s_2 \quad (\text{since } b_4 \leq b_3)$$

$$\vdots$$

$$s_{2n} = s_{2n-2} + (b_{2n-1} - b_{2n}) \geq s_{2n-2} \quad (\text{since } b_{2n} \leq b_{2n-1})$$

This shows $\{s_{2n}\}$ is increasing. Moreover,

$$s_{2n} = b_1 - (b_2 - b_3) - (b_4 - b_5) - \cdots - (b_{2n-2} - b_{2n-1}) - b_{2n}$$

Every term in parentheses is positive, so $s_{2n} \leq b_1$. This shows $\{s_{2n}\}$ is bounded above.

- What can we conclude about the limit of the even partial sums $s_2, s_4, s_6, \ldots$?

Since $\{s_{2n}\}$ is increasing and bounded above, it converges by the Monotonic Sequence Theorem. Let $\lim_{n\to\infty} s_{2n} = s$.

- Why do the odd partial sums $s_1, s_3, s_5, \ldots$ converge to the same limit s?

Note: $s_{2n+1} = s_{2n} + b_{2n+1}$

0

As $n \to \infty$, $\lim_{n\to\infty} s_{2n+1} = \lim_{n\to\infty} s_{2n} + \lim_{n\to\infty} b_{2n+1}$

$$= s + 0$$

$$= s$$

- What does this imply about the convergence of the series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$?

Since both the even partial sums $\{s_{2n}\}$ and the odd partial sums $\{s_{2n+1}\}$ converge to the same limit s, the full sequence of partial sums $\{s_n\}$ also converges to s.

Zipper

Example. Determine whether the alternating harmonic series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1}{n}$$

$$b_n = \frac{1}{n}$$

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

converges or diverges.

Check the conditions for the Alternating Series Test:

① $b_{n+1} < b_n$ because $\frac{1}{n+1} < \frac{1}{n}$ ✓

② $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0$ ✓

The series converges by the Alternating Series Test

Example. Determine whether the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1}$$

converges or diverges.

Try the Alternating Series Test:

the terms are $b_n = \frac{3n}{4n-1}$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{3n}{4n-1} \cdot \frac{1/n}{1/n} = \lim_{n\to\infty} \frac{3}{4-\frac{1}{n}} = \frac{3}{4-0} = \frac{3}{4}$$

Since $\lim_{n\to\infty} b_n \neq 0$, the Alternating Series Test does not apply

Instead, consider $a_n = \frac{(-1)^n 3n}{4n-1}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^n \cdot \frac{3n}{4n-1}$$

This limit does not exist because $(-1)^n$ oscillates. The series diverges by the Test for Divergence.

Example. Test the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{n^3+1}$ for convergence or divergence.

Check the conditions for the Alternating Series Test:

(1) b_n is decreasing: Consider $f(x) = \frac{x^2}{x^3+1}$

$$f'(x) = \frac{(x^3+1)\cdot(2x) - x^2(3x^2)}{(x^3+1)^2} = \frac{x(2-x^3)}{(x^3+1)^2}$$

$f'(x) < 0$ when $2-x^3 < 0$, i.e. when $x > \sqrt[3]{2}$

Hence $f(x)$ is decreasing on $(\sqrt[3]{2}, \infty)$ and b_n is eventually decreasing for $n \geq 2$.

(2) $\lim_{n\to\infty} b_n = 0$.

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n^2}{n^3+1} \cdot \frac{1/n^2}{1/n^2} = \lim_{n\to\infty} \frac{1}{n+\frac{1}{n^2}} = 0$$

The series converges by the Alternating Series Test.

Estimating Sums of Alternating Series

A partial sum s_n of any convergent series can be used as an approximation to the total sum s. However, this is only useful if we can estimate the accuracy of the approximation. The error involved in approximating s by s_n is called the remainder:

$$R_n = s - s_n.$$

The following theorem provides a bound for the size of this error for series that satisfy the conditions of the Alternating Series Test. Specifically, the error is smaller than b_{n+1}, the absolute value of the first neglected term.

Theorem. If $s = \sum_{n=1}^{\infty}(-1)^{n-1}b_n$, where $b_n > 0$, is the sum of an alternating series that satisfies:

(i) $b_{n+1} \leq b_n$

(ii) $\lim_{n\to\infty} b_n = 0$,

then the remainder $R_n = s - s_n$ satisfies:

$$|R_n| = |s - s_n| \leq b_{n+1},$$

where b_{n+1} is the absolute value of the first neglected term.

Proof.

From the Alternating Series Test, the total sum s lies between any two consecutive partial sums. So, either:

$$s_{n+1} \leq s \leq s_n \qquad \text{OR} \qquad s_n \leq s \leq s_{n+1}$$

Indeed, we showed that s is larger than all even partial sums and that s is smaller than all odd partial sums.

Hence $|s-s_n| \leq |s_n - s_{n+1}| = b_{n+1}$

We conclude $|R_n| = |s-s_n| \leq b_{n+1}$

□

Example. Find the sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ correct to three decimal places.

① Verify convergence by A.S.T.

(i) $b_{n+1} = \frac{1}{(n+1)!} < \frac{1}{n!} = b_n$ ✓

(ii) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n!} = 0$ ✓

② Write out the first few terms:

$$s = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \cdots$$

$$= 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} - \frac{1}{5040} + \cdots$$

Note: $b_7 = \frac{1}{5040} < \frac{1}{5000} = 0.0002$

③ Approximate the sum:

Calculator

$$s_6 = 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} \approx 0.368056$$

④ Estimate the error (how close is this to the actual sum?)

$$|R_6| = |s - s_6| \leq b_7 = \frac{1}{7!} = \frac{1}{5040} \approx 0.0002$$

⑤ Conclusion: $s \approx 0.368056$ with error bounded by 0.0002,

so 0.368 is correct to 3 decimal places.

Remark. The rule that the error $|s - s_n|$ is smaller than the first neglected term b_{n+1} is valid only for alternating series that satisfy the conditions of the Alternating Series Estimation Theorem. This rule does not apply to other types of series.

Absolute Convergence and Conditional Convergence

Definition. Given any series $\sum a_n$, we can consider the corresponding series:

$$\sum |a_n| = |a_1| + |a_2| + |a_3| + \cdots ,$$

whose terms are the absolute values of the terms of the original series.

A series $\sum a_n$ is called **absolutely convergent** if the series of absolute values $\sum |a_n|$ is convergent.

Notice that if $\sum a_n$ is a series with positive terms, then $|a_n| = a_n$, and so absolute convergence is the same as convergence in this case.

Example. Determine whether the alternating series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

is absolutely convergent.

Consider the series of absolute values:

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

This is a p-series with $p = 2 > 1$, which converges.

Therefore, $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ is absolutely convergent.

Definition. A series $\sum a_n$ is called **conditionally convergent** if it is convergent but not absolutely convergent; that is, if $\sum a_n$ converges but $\sum |a_n|$ diverges.

Example. Show that the that the alternating harmonic series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

is conditionally convergent.

We showed this converges by the Alternating Series Test

Consider the series of absolute values: $\sum_{n=1}^{\infty} \left| (-1)^{n-1} \frac{1}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$

This is the harmonic series, which is a p-series with $p=1$, and it diverges.

Remark. This example shows that it is possible for a series to be convergent but not absolutely convergent. The following theorem states that absolute convergence implies convergence.

Theorem. If a series $\sum a_n$ is absolutely convergent, then it is convergent.

- We have $0 \le a_n + |a_n| \le 2|a_n|$ since $|a_n|$ is either a_n or $-a_n$
- If $\sum a_n$ is absolutely convergent, then $\sum |a_n|$ converges.
  In particular, $\sum 2|a_n|$ converges.
- By the D.C.T., $\sum a_n + |a_n|$ converges
- But now $\sum a_n = \sum (a_n + |a_n|) - \sum |a_n|$ is a difference of two convergent series, which is convergent.

$\square$

Example. Determine whether the series

$$\sum_{n=1}^{\infty} \frac{\cos n}{n^2} = \frac{\cos 1}{1^2} + \frac{\cos 2}{2^2} + \frac{\cos 3}{3^2} + \cdots$$

is convergent or divergent.

The series has both positive and negative terms, but it is not alternating.

To analyze convergence, consider the series of absolute values

$$\sum_{n=1}^{\infty} \left| \frac{\cos n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{|\cos n|}{n^2} \leq \sum_{n=1}^{\infty} \frac{1}{n^2} \qquad (\text{since } |\cos n| \leq 1)$$

The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series $(p = 2 > 1)$

By the D.C.T., $\sum_{n=1}^{\infty} \frac{|\cos n|}{n^2}$ converges.

Therefore, $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ is absolutely convergent, and thus convergent.

Example. Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$ is absolutely convergent, conditionally convergent, or divergent.

- Test for absolute convergence : $\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^3} \right| = \sum_{n=1}^{\infty} \frac{1}{n^3}$
- This is a p-series with $p = 3 > 1$, which is convergent
- Hence, the given series is absolutely convergent

Example. Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}$ is absolutely convergent, conditionally convergent, or divergent.

- Test for absolute convergence : $\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{\sqrt[3]{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$
- This is a p-series with $p = \frac{1}{3} \leq 1$, which diverges.
- Test for convergence using the Alternating Series Test with $b_n = \frac{1}{\sqrt[3]{n}}$

  Since $b_{n+1} \leq b_n$ and $\lim_{n\to\infty} b_n = 0$, this converges by A.S.T.
- Conditionally convergent.

Example. Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^n n}{2n+1}$ is absolutely convergent, conditionally convergent, or divergent.

- This series is alternating, but $\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^n \frac{n}{2n+1}$
- Observing the magnitude of a_n, we find $\lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2}$
- Since $\lim_{n\to\infty} a_n \neq 0$ (the terms oscillate between approx $\pm \frac{1}{2}$)

  the series diverges by the test for divergence.