§11.4: Comparison Tests Practice

Directions: For each series below, choose an appropriate comparison series, decide whether to use the Comparison Test or the Limit Comparison Test, and determine whether the series converges or diverges. Your response should include an introduction in which you verify any conditions, a clear application of the test, and a formal conclusion.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{\arctan n}{2^n}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{\arctan n}{\sqrt{n}}$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{e^n + n^2}$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{e^n - n^2}$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{3n-2}{2n^3+5}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2-n+5}{n^3-3n+6}$

§11.4: Comparison Tests Practice — Solutions

1. $\displaystyle\sum_{n=1}^{\infty} \frac{\arctan n}{2^n}$

   (a) **Introduction:** Let

   $$a_n = \frac{\arctan n}{2^n} \qquad \text{and} \qquad b_n = \frac{\pi/2}{2^n}.$$

   Both $\sum a_n$ and $\sum b_n$ have positive terms, so the Direct Comparison Test applies.

   (b) **Apply D.C.T.:** Since $\arctan n \leq \dfrac{\pi}{2}$, we have

   $$\frac{\arctan n}{2^n} \leq \frac{\pi/2}{2^n}.$$

   Also,

   $$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{\pi/2}{2^n} = \sum_{n=1}^{\infty} \frac{\pi}{2}\left(\frac{1}{2}\right)^n,$$

   is a geometric series with $|r| = \frac{1}{2} < 1$, so $\sum b_n$ converges.

   (c) **Conclusion:** Since $a_n \leq b_n$ and $\sum b_n$ converges,

   $$\sum_{n=1}^{\infty} \frac{\arctan n}{2^n}$$

   converges by the Direct Comparison Test.

2. $\displaystyle\sum_{n=1}^{\infty} \frac{\arctan n}{\sqrt{n}}$

(a) **Introduction:** Let

$$a_n = \frac{\arctan n}{\sqrt{n}} \qquad \text{and} \qquad b_n = \frac{\pi/4}{\sqrt{n}}.$$

Both $\sum a_n$ and $\sum b_n$ have positive terms, so the Direct Comparison Test applies.

(b) **Apply D.C.T.:** Since $\arctan n \geq \arctan(1) = \frac{\pi}{4}$, we have

$$\frac{\arctan n}{\sqrt{n}} \geq \frac{\pi/4}{\sqrt{n}}.$$

Also,

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{\pi/4}{\sqrt{n}} = \frac{\pi}{4} \sum_{n=1}^{\infty} \frac{1}{n^{1/2}},$$

and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ is a p-series with $p = \frac{1}{2} \leq 1$. Therefore, $\sum b_n$ diverges.

(c) **Conclusion:** Since $a_n \geq b_n$ and $\sum b_n$ diverges,

$$\sum_{n=1}^{\infty} \frac{\arctan n}{\sqrt{n}}$$

diverges by the Direct Comparison Test.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{e^n + n^2}$

(a) **Introduction:** Let

$$a_n = \frac{1}{e^n + n^2} \qquad \text{and} \qquad b_n = \frac{1}{e^n}.$$

Both $\sum a_n$ and $\sum b_n$ have positive terms, so the Direct Comparison Test applies.

(b) **Apply D.C.T.:** Since

$$e^n + n^2 \geq e^n,$$

we have

$$\frac{1}{e^n + n^2} \leq \frac{1}{e^n}.$$

Also,

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{e^n} = \sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^n,$$

is a geometric series with $|r| = \frac{1}{e} < 1$, so $\sum b_n$ converges.

(c) **Conclusion:** Since $a_n \leq b_n$ and $\sum b_n$ converges,

$$\sum_{n=1}^{\infty} \frac{1}{e^n + n^2}$$

converges by the Direct Comparison Test.

4. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{e^n - n^2}$

(a) **Introduction:** Let

$$a_n = \frac{1}{e^n - n^2} \qquad \text{and} \qquad b_n = \frac{1}{e^n}.$$

Both $\sum a_n$ and $\sum b_n$ have positive terms, so the Limit Comparison Test applies.

(b) **Apply L.C.T.:**

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{e^n - n^2}}{\frac{1}{e^n}} = \lim_{n\to\infty} \frac{e^n}{e^n - n^2} = \lim_{n\to\infty} \frac{1}{1 - \frac{n^2}{e^n}} = 1.$$

Also,

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{e^n} = \sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^n,$$

is a geometric series with $|r| = \frac{1}{e} < 1$, so $\sum b_n$ converges.

(c) **Conclusion:** Since

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1 > 0$$

and $\sum b_n$ converges,

$$\sum_{n=1}^{\infty} \frac{1}{e^n - n^2}$$

converges by the Limit Comparison Test.

5. $\displaystyle\sum_{n=1}^{\infty} \frac{3n-2}{2n^3+5}$

(a) **Introduction:** Let

$$a_n = \frac{3n-2}{2n^3+5} \qquad \text{and} \qquad b_n = \frac{3}{2n^2}.$$

Both $\sum a_n$ and $\sum b_n$ have positive terms, so the Direct Comparison Test applies.

(b) **Apply D.C.T.:** Since

$$3n-2 \leq 3n \qquad \text{and} \qquad 2n^3+5 \geq 2n^3,$$

we have

$$\frac{3n-2}{2n^3+5} \leq \frac{3n}{2n^3} = \frac{3}{2n^2}.$$

Also,

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{3}{2n^2} = \sum_{n=1}^{\infty} \frac{3}{2} \cdot \frac{1}{n^2},$$

is a constant multiple of a p-series with $p = 2 > 1$, so $\sum b_n$ converges.

(c) **Conclusion:** Since $a_n \leq b_n$ and $\sum b_n$ converges,

$$\sum_{n=1}^{\infty} \frac{3n-2}{2n^3+5}$$

converges by the Direct Comparison Test.

6. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 - n + 5}{n^3 - 3n + 6}$

(a) **Introduction:** Let

$$a_n = \frac{n^2 - n + 5}{n^3 - 3n + 6} \qquad \text{and} \qquad b_n = \frac{1}{n}.$$

Both $\sum a_n$ and $\sum b_n$ have positive terms, so the Limit Comparison Test applies.

(b) **Apply L.C.T.:**

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n^2-n+5}{n^3-3n+6}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^3 - n^2 + 5n}{n^3 - 3n + 6} = \lim_{n\to\infty} \frac{1 - \frac{1}{n} + \frac{5}{n^2}}{1 - \frac{3}{n^2} + \frac{6}{n^3}} = 1.$$

Also,

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n},$$

is the harmonic series, so $\sum b_n$ diverges.

(c) **Conclusion:** Since

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1 > 0$$

and $\sum b_n$ diverges,

$$\sum_{n=1}^{\infty} \frac{n^2 - n + 5}{n^3 - 3n + 6}$$

diverges by the Limit Comparison Test.