Direct Comparison Test

Direct Comparison Test. Let $a_n \geq 0$ and $b_n \geq 0$ for all sufficiently large n.

- If $a_n \leq b_n$ for all sufficiently large n, and if $\sum b_n$ converges, then $\sum a_n$ also converges.
- If $b_n \leq a_n$ for all sufficiently large n, and if $\sum b_n$ diverges, then $\sum a_n$ also diverges.

This test is most useful when a series can be compared to a simpler series whose behavior is already known.

Use the Direct Comparison Test to determine whether each series converges or diverges by comparing it to a known p-series or geometric series.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{3^n+5}$
3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+n}$
4. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^3+1}$
5. $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2+2}{n^4+n^2+1}$
6. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{4^n+n}$
7. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+2}$
8. $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{3^n+2^n}$
9. $\displaystyle\sum_{n=1}^{\infty} \frac{1+\sin^2\left(\frac{1}{n}\right)}{n}$
10. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

Limit Comparison Test

The **Limit Comparison Test** is used to compare two positive-term series

$$\sum a_n \qquad \text{and} \qquad \sum b_n.$$

Compute

$$L = \lim_{n\to\infty} \frac{a_n}{b_n}.$$

If

$$0 < L < \infty,$$

then $\sum a_n$ and $\sum b_n$ either both converge or both diverge. In practice, the Limit Comparison Test is useful when a_n and b_n have the same dominant behavior for large n, even if one is not always larger than the other.

Use the Limit Comparison Test to determine whether each series converges or diverges.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+3n}{n^3-4}$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^4-1}$
3. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{n^2+5}$
4. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$
5. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+\sqrt{n+1}}$
6. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^4+3n}}$
7. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n+2^n}{4^n+1}$
8. $\displaystyle\sum_{n=1}^{\infty} \frac{5}{n^2+(-1)^n}$
9. $\displaystyle\sum_{n=1}^{\infty} \frac{4}{n^2+\tan^{-1}(n)}$
10. $\displaystyle\sum_{n=1}^{\infty} \frac{5}{n^2+(-1)^n}$