Direct Comparison Test

1. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

   Let
   $$a_n = \frac{1}{n^2+1}.$$
   For all $n \geq 1$,
   $$0 < a_n = \frac{1}{n^2+1} \leq \frac{1}{n^2}.$$
   The series
   $$\sum_{n=1}^{\infty} \frac{1}{n^2}$$
   is a p-series with $p = 2 > 1$, so it converges. Therefore, by the Direct Comparison Test,
   $$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$$
   converges.

   Conclusion: The series **converges**.

2. $\sum_{n=1}^{\infty} \frac{2^n}{3^n+5}$

   Let
   $$a_n = \frac{2^n}{3^n+5}.$$
   For all $n \geq 1$,
   $$0 < a_n = \frac{2^n}{3^n+5} \leq \frac{2^n}{3^n} = \left(\frac{2}{3}\right)^n.$$
   The series
   $$\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$$
   is geometric with common ratio $r = \frac{2}{3}$, and since $|r| < 1$, it converges. Therefore, by the Direct Comparison Test,
   $$\sum_{n=1}^{\infty} \frac{2^n}{3^n+5}$$
   converges.

   Conclusion: The series **converges**.

3. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+n}$

   Let
   $$a_n = \frac{1}{\sqrt{n}+n}.$$
   For $n \geq 1$, we have $\sqrt{n} \leq n$, so
   $$\sqrt{n} + n \leq 2n.$$

Taking reciprocals gives

$$a_n = \frac{1}{\sqrt{n}+n} \geq \frac{1}{2n}.$$

The series

$$\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges because the harmonic series diverges. Therefore, by the Direct Comparison Test,

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+n}$$

diverges.

Conclusion: The series **diverges**.

4. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^3+1}$

Let

$$a_n = \frac{n}{n^3+1}.$$

For all $n \geq 1$,

$$0 < a_n = \frac{n}{n^3+1} \leq \frac{n}{n^3} = \frac{1}{n^2}.$$

The series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

is a p-series with $p = 2 > 1$, so it converges. Therefore, by the Direct Comparison Test,

$$\sum_{n=1}^{\infty} \frac{n}{n^3+1}$$

converges.

Conclusion: The series **converges**.

5. $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2+2}{n^4+n^2+1}$

Let

$$a_n = \frac{3n^2+2}{n^4+n^2+1}.$$

For $n \geq 1$, we have $2 \leq 2n^2$, so $3n^2+2 \leq 5n^2$. Also,

$$n^4+n^2+1 \geq n^4.$$

Hence, for $n \geq 1$,

$$0 < a_n = \frac{3n^2+2}{n^4+n^2+1} \leq \frac{5n^2}{n^4} = \frac{5}{n^2}.$$

The series

$$\sum_{n=1}^{\infty} \frac{5}{n^2} = 5\sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges because $\sum \frac{1}{n^2}$ is a p-series with $p = 2 > 1$. Therefore, by the Direct Comparison Test,

$$\sum_{n=1}^{\infty} \frac{3n^2+2}{n^4+n^2+1}$$

converges.

Conclusion: The series **converges**.

6. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{4^n+n}$

Let

$$a_n = \frac{2^n}{4^n+n}.$$

For all $n \geq 1$,

$$0 < a_n = \frac{2^n}{4^n+n} \leq \frac{2^n}{4^n} = \left(\frac{1}{2}\right)^n.$$

The series

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$$

is geometric with common ratio $r = \frac{1}{2}$, and since $|r| < 1$, it converges. Therefore, by the Direct Comparison Test,

$$\sum_{n=1}^{\infty} \frac{2^n}{4^n+n}$$

converges.

Conclusion: The series **converges**.

7. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+2}$

Let

$$a_n = \frac{n}{n^2+2}.$$

For $n \geq 2$, we have $2 \leq n^2$, so

$$n^2 + 2 \leq 2n^2.$$

Therefore, for $n \geq 2$,

$$a_n = \frac{n}{n^2+2} \geq \frac{n}{2n^2} = \frac{1}{2n}.$$

The series

$$\sum_{n=2}^{\infty} \frac{1}{2n} = \frac{1}{2}\sum_{n=2}^{\infty} \frac{1}{n}$$

diverges because the harmonic series diverges. Therefore, by the Direct Comparison Test,

$$\sum_{n=1}^{\infty} \frac{n}{n^2+2}$$

diverges.

Conclusion: The series **diverges**.

8. $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{3^n + 2^n}$

Let

$$a_n = \frac{5^n}{3^n + 2^n}.$$

Since $2^n \leq 3^n$ for all $n \geq 1$, we have

$$3^n + 2^n \leq 2 \cdot 3^n.$$

Thus, for all $n \geq 1$,

$$a_n = \frac{5^n}{3^n + 2^n} \geq \frac{5^n}{2 \cdot 3^n} = \frac{1}{2}\left(\frac{5}{3}\right)^n.$$

The series

$$\sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{5}{3}\right)^n$$

is geometric with common ratio $r = \frac{5}{3}$, and since $r > 1$, it diverges. Therefore, by the Direct Comparison Test,

$$\sum_{n=1}^{\infty} \frac{5^n}{3^n + 2^n}$$

diverges.

Conclusion: The series **diverges**.

9. $\displaystyle\sum_{n=1}^{\infty} \frac{1 + \sin^2\left(\frac{1}{n}\right)}{n}$

Let

$$a_n = \frac{1 + \sin^2\left(\frac{1}{n}\right)}{n}.$$

Since $\sin^2\left(\frac{1}{n}\right) \geq 0$ for all $n \geq 1$, we have

$$a_n = \frac{1 + \sin^2\left(\frac{1}{n}\right)}{n} \geq \frac{1}{n}.$$

The series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

is the harmonic series, which diverges. Therefore, by the Direct Comparison Test,

$$\sum_{n=1}^{\infty} \frac{1 + \sin^2\left(\frac{1}{n}\right)}{n}$$

diverges.

Conclusion: The series **diverges**.

10. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

Let

$$a_n = \frac{n!}{n^n}.$$

For $n \geq 2$,

$$a_n = \frac{1}{n} \cdot \frac{2}{n} \cdot \frac{3}{n} \cdots \frac{n-1}{n} \cdot \frac{n}{n}.$$

Each factor is at most 1, so

$$a_n \le \frac{1}{n} \cdot \frac{2}{n} \cdot 1 \cdot 1 \cdots 1 = \frac{2}{n^2}.$$

Thus, for $n \ge 2$,

$$0 < a_n \le \frac{2}{n^2}.$$

The series

$$\sum_{n=2}^{\infty} \frac{2}{n^2} = 2 \sum_{n=2}^{\infty} \frac{1}{n^2}$$

converges because $\sum \frac{1}{n^2}$ is a p-series with $p = 2 > 1$. Therefore, by the Direct Comparison Test,

$$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$

converges.

Conclusion: The series **converges**.

Limit Comparison Test

1. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+3n}{n^3-4}$

   Let

   $$a_n = \frac{n^2+3n}{n^3-4} \qquad \text{and} \qquad b_n = \frac{1}{n}.$$

   For $n \geq 2$, both a_n and b_n are positive. Compute

   $$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n^2+3n}{n^3-4}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^3+3n^2}{n^3-4}.$$

   Divide numerator and denominator by n^3:

   $$\lim_{n\to\infty} \frac{n^3+3n^2}{n^3-4} = \lim_{n\to\infty} \frac{1+\frac{3}{n}}{1-\frac{4}{n^3}} = 1.$$

   Since $0 < 1 < \infty$, the Limit Comparison Test applies. Because

   $$\sum_{n=1}^{\infty} \frac{1}{n}$$

   is the harmonic series and diverges, it follows that

   $$\sum_{n=1}^{\infty} \frac{n^2+3n}{n^3-4}$$

   also diverges.

   Conclusion: The series **diverges**.

2. $\displaystyle\sum_{n=2}^{\infty} \frac{n^2}{n^4-1}$

   Let

   $$a_n = \frac{n^2}{n^4-1} \qquad \text{and} \qquad b_n = \frac{1}{n^2}.$$

   For $n \geq 2$, both a_n and b_n are positive. Compute

   $$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n^2}{n^4-1}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^4}{n^4-1}.$$

   Divide numerator and denominator by n^4:

   $$\lim_{n\to\infty} \frac{n^4}{n^4-1} = \lim_{n\to\infty} \frac{1}{1-\frac{1}{n^4}} = 1.$$

   Since $0 < 1 < \infty$, the Limit Comparison Test applies. Because

   $$\sum_{n=2}^{\infty} \frac{1}{n^2}$$

is a p-series with $p = 2 > 1$, it converges. Therefore,

$$\sum_{n=2}^{\infty} \frac{n^2}{n^4 - 1}$$

also converges.

Conclusion: The series **converges**.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{n^2+5}$

Let

$$a_n = \frac{\sqrt{n}+1}{n^2+5} \qquad \text{and} \qquad b_n = \frac{1}{n^{3/2}}.$$

Both a_n and b_n are positive for all $n \geq 1$. Compute

$$\begin{aligned} \lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\frac{\sqrt{n}+1}{n^2+5}}{\frac{1}{n^{3/2}}} \\ &= \lim_{n\to\infty} \frac{n^{3/2}(\sqrt{n}+1)}{n^2+5} \\ &= \lim_{n\to\infty} \frac{n^2+n^{3/2}}{n^2+5}. \end{aligned}$$

Divide numerator and denominator by n^2:

$$\begin{aligned} \lim_{n\to\infty} \frac{n^2+n^{3/2}}{n^2+5} &= \lim_{n\to\infty} \frac{1+\frac{1}{\sqrt{n}}}{1+\frac{5}{n^2}} \\ &= 1. \end{aligned}$$

Since $0 < 1 < \infty$, the Limit Comparison Test applies. Because

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

is a p-series with $p = \frac{3}{2} > 1$, it converges. Therefore,

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{n^2+5}$$

also converges.

Conclusion: The series **converges**.

4. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$

Let

$$a_n = \frac{1}{n+\sqrt{n}} \qquad \text{and} \qquad b_n = \frac{1}{n}.$$

Both a_n and b_n are positive for all $n \geq 1$. Compute

$$\begin{aligned} \lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\frac{1}{n+\sqrt{n}}}{\frac{1}{n}} \\ &= \lim_{n\to\infty} \frac{n}{n+\sqrt{n}}. \end{aligned}$$

Divide numerator and denominator by n:

$$\lim_{n\to\infty} \frac{n}{n+\sqrt{n}} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{\sqrt{n}}}$$
$$= 1.$$

Since $0 < 1 < \infty$, the Limit Comparison Test applies. Because

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

is the harmonic series and diverges, it follows that

$$\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$$

also diverges.

Conclusion: The series **diverges**.

5. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+\sqrt{n+1}}$

Let

$$a_n = \frac{1}{\sqrt{n}+\sqrt{n+1}} \qquad \text{and} \qquad b_n = \frac{1}{\sqrt{n}}.$$

Both a_n and b_n are positive for all $n \geq 1$. Compute

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}+\sqrt{n+1}}}{\frac{1}{\sqrt{n}}}$$
$$= \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}+\sqrt{n+1}}$$
$$= \lim_{n\to\infty} \frac{1}{1+\sqrt{1+\frac{1}{n}}}$$
$$= \frac{1}{2}.$$

Since $0 < \frac{1}{2} < \infty$, the Limit Comparison Test applies. Because

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

is a p-series with $p = \frac{1}{2} < 1$, it diverges. Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+\sqrt{n+1}}$$

also diverges.

Conclusion: The series **diverges**.

6. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^4+3n}}$

Let

$$a_n = \frac{1}{\sqrt{n^4+3n}} \qquad \text{and} \qquad b_n = \frac{1}{n^2}.$$

Both a_n and b_n are positive for all $n \geq 1$. Compute

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n^4+3n}}}{\frac{1}{n^2}}$$
$$= \lim_{n\to\infty} \frac{n^2}{\sqrt{n^4+3n}}.$$

Factor n^4 from inside the square root:

$$\frac{n^2}{\sqrt{n^4+3n}} = \frac{n^2}{\sqrt{n^4\left(1+\frac{3}{n^3}\right)}}$$
$$= \frac{n^2}{n^2\sqrt{1+\frac{3}{n^3}}}$$
$$= \frac{1}{\sqrt{1+\frac{3}{n^3}}}.$$

Therefore,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{3}{n^3}}} = 1.$$

Since $0 < 1 < \infty$, the Limit Comparison Test applies. Because

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

is a p-series with $p = 2 > 1$, it converges. Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^4+3n}}$$

also converges.

Conclusion: The series **converges**.

7. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n+2^n}{4^n+1}$

Let

$$a_n = \frac{3^n+2^n}{4^n+1} \qquad \text{and} \qquad b_n = \left(\frac{3}{4}\right)^n.$$

Both a_n and b_n are positive for all $n \geq 1$. Compute

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{3^n+2^n}{4^n+1}}{\left(\frac{3}{4}\right)^n}$$
$$= \lim_{n\to\infty} \frac{3^n+2^n}{4^n+1} \cdot \frac{4^n}{3^n}$$
$$= \lim_{n\to\infty} \frac{4^n\left(3^n+2^n\right)}{3^n(4^n+1)}$$
$$= \lim_{n\to\infty} \frac{1+\left(\frac{2}{3}\right)^n}{1+\frac{1}{4^n}}$$
$$= 1.$$

Since $0 < 1 < \infty$, the Limit Comparison Test applies. Because

$$\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n$$

is geometric with common ratio $r = \frac{3}{4}$, and since $|r| < 1$, it converges. Therefore,

$$\sum_{n=1}^{\infty} \frac{3^n + 2^n}{4^n + 1}$$

also converges.

Conclusion: The series **converges**.

8. $\displaystyle\sum_{n=1}^{\infty} \frac{5}{n^2 + (-1)^n}$

Let

$$a_n = \frac{5}{n^2 + (-1)^n} \qquad \text{and} \qquad b_n = \frac{1}{n^2}.$$

For $n \geq 2$, both a_n and b_n are positive. Compute

$$\begin{aligned} \lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\frac{5}{n^2+(-1)^n}}{\frac{1}{n^2}} \\ &= \lim_{n\to\infty} \frac{5n^2}{n^2 + (-1)^n}. \end{aligned}$$

Divide numerator and denominator by n^2:

$$\begin{aligned} \lim_{n\to\infty} \frac{5n^2}{n^2 + (-1)^n} &= \lim_{n\to\infty} \frac{5}{1 + \frac{(-1)^n}{n^2}} \\ &= 5. \end{aligned}$$

Since $0 < 5 < \infty$, the Limit Comparison Test applies. Because

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

is a p-series with $p = 2 > 1$, it converges. Therefore,

$$\sum_{n=1}^{\infty} \frac{5}{n^2 + (-1)^n}$$

also converges.

Conclusion: The series **converges**.

9. $\displaystyle\sum_{n=1}^{\infty} \frac{4}{n^2 + \tan^{-1}(n)}$

Let

$$a_n = \frac{4}{n^2 + \tan^{-1}(n)} \qquad \text{and} \qquad b_n = \frac{1}{n^2}.$$

Both a_n and b_n are positive for all $n \geq 1$. Compute

$$\begin{aligned} \lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\frac{4}{n^2+\tan^{-1}(n)}}{\frac{1}{n^2}} \\ &= \lim_{n\to\infty} \frac{4n^2}{n^2 + \tan^{-1}(n)}. \end{aligned}$$

Divide numerator and denominator by n^2:

$$\lim_{n\to\infty} \frac{4n^2}{n^2 + \tan^{-1}(n)} = \lim_{n\to\infty} \frac{4}{1 + \frac{\tan^{-1}(n)}{n^2}}$$
$$= 4.$$

Since $0 < 4 < \infty$, the Limit Comparison Test applies. Because

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

is a p-series with $p = 2 > 1$, it converges. Therefore,

$$\sum_{n=1}^{\infty} \frac{4}{n^2 + \tan^{-1}(n)}$$

also converges.

Conclusion: The series **converges**.