11.4 The Comparison Tests

In the comparison tests the idea is to compare a given series with a series that is known to be convergent or divergent. If two series have only positive terms, we can compare corresponding terms directly to see which are larger (the Direct Comparison Test) or we can investigate the limit of the ratios of corresponding terms (the Limit Comparison Test).

The Direct Comparison Test

Theorem. Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

1. If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.
2. If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is also divergent.

Proof. Convergence : $s_n = \sum_{i=1}^{n} a_i$, $t_n = \sum_{i=1}^{n} b_i$, $t = \sum_{n=1}^{\infty} b_i$

- Both $\{s_n\}$ and $\{t_n\}$ are increasing sequences because the terms are positive
- Since $a_i \leq b_i$ we have $s_n \leq t_n < t$
- Thus $\{s_n\}$ is increasing and bounded above, so it converges by the Monotonic Sequence Theorem. By definition, $\sum a_n$ convergences.

Divergence: Since $a_i \geq b_i$ we have $s_n \geq t_n$

- If $\sum b_n$ is divergent, then $t_n \to \infty$ (since t_n is increasing)
- Hence $s_n \to \infty$ and $\sum a_n$ diverges

□

When using Direct Comparison Test, we must have some known series $\sum b_n$ for the purpose of comparison. Most of the time we use one of these series:

- A p-series $\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.
- A geometric series $\sum ar^{n-1}$ converges if $|r| < 1$ and diverges if $|r| \geq 1$.

Example. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$ converges or diverges.

- For large n, the dominant term in the denominator is $2n^2$.
- Compare the given series to $\sum \frac{5}{2n^2}$:

$$\underbrace{\frac{5}{2n^2+4n+3}}_{\text{Bigger denominator}} < \frac{5}{2n^2}$$

- The series $\displaystyle\sum_{n=1}^{\infty} \frac{5}{2n^2} = \frac{5}{2}\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges because $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a p-series with $p=2$ and $2>1$.
- The series $\displaystyle\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$ converges by the Direct Comparison Test.

Remark. Although the condition $a_n \le b_n$ or $a_n \ge b_n$ in the Direct Comparison Test is given for all n, we need verify only that it holds for $n \ge N$, where N is some fixed integer, because the convergence of a series is not affected by a finite number of terms.

Example. Test the series $\displaystyle\sum_{k=1}^{\infty} \frac{\ln k}{k}$ for convergence or divergence.

- We used the Integral Test to test this in §11.3, but we can also compare it to the harmonic series.
- Observe that $\ln k > 1$ for $k \ge 3$ and so $\frac{\ln k}{k} > \frac{1}{k}$ for $k \ge 3$
- We know that $\sum \frac{1}{k}$ is divergent (p-series with $p=1$)
- By the Direct Comparison Test, $\displaystyle\sum_{k=1}^{\infty} \frac{\ln k}{k}$ is also divergent.

Remark. The Direct Comparison Test is conclusive only if the terms of the series being tested are smaller than those of a convergent series or larger than those of a divergent series. If the terms are larger than the terms of a convergent series or smaller than those of a divergent series, then the Direct Comparison Test doesn't apply. Consider, for instance, the series:

$$\sum_{n=1}^{\infty} \frac{1}{2^n - 1}.$$

The inequality:

$$\frac{1}{2^n - 1} > \frac{1}{2^n},$$

is useless as far as the Direct Comparison Test is concerned because $\sum b_n = \sum \left(\frac{1}{2}\right)^n$ is convergent and $a_n > b_n$. Nonetheless, we have the feeling that $\sum \frac{1}{2^n - 1}$ ought to be convergent because it is very similar to the convergent geometric series $\sum \left(\frac{1}{2}\right)^n$.

The Limit Comparison Test

Theorem. Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c,$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

Proof.

- Let m and M be positive numbers such that $m < c < M$
- Because $\frac{a_n}{b_n}$ is close to c for large n, there is an integer N such that:

$$m < \frac{a_n}{b_n} < M \quad \text{when} \quad n > N$$

- Hence $m \cdot b_n < a_n < M \cdot b_n$ when $n > N$
- If $\sum b_n$ converges, so does $\sum M \cdot b_n$. Thus $\sum a_n$ converges by Direct Comparison Test
- If $\sum b_n$ diverges, so does $\sum m \cdot b_n$. Thus $\sum a_n$ diverges by Direct Comparison Test

$\square$

Example. Test the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ for convergence or divergence.

Use the limit comparison test with $a_n = \dfrac{1}{2^n - 1}$ and $b_n = \dfrac{1}{2^n}$

Then $$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{2^n-1}}{\frac{1}{2^n}} = \lim_{n\to\infty} \frac{2^n}{2^n - 1} \cdot \frac{1/2^n}{1/2^n} = \lim_{n\to\infty} \frac{1}{1 - \frac{1}{2^n}}$$

$$= \frac{1}{1-0} = 1$$

Since $\sum \frac{1}{2^n}$ is a convergent geometric series (since $|r| = |\frac{1}{2}| < 1$)

$\sum \frac{1}{2^n - 1}$ also converges by the Limit Comparison Test.

Example. Determine whether the series $\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^5}}$ converges or diverges.

- The dominant part of the numerator is $2n^2$. The dominant part of the denominator is $\sqrt{n^5} = n^{5/2}$

- This suggests taking $a_n = \frac{2n^2+3n}{\sqrt{5+n^5}}$, $b_n = \frac{2n^2}{n^{5/2}} = \frac{2}{n^{1/2}}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\dfrac{2n^2+3n}{\sqrt{5+n^5}}}{\dfrac{2}{n^{1/2}}} = \lim_{n\to\infty} \frac{2n^2+3n}{\sqrt{5+n^5}} \cdot \frac{n^{1/2}}{2}$$

$$= \lim_{n\to\infty} \frac{2n^{5/2}+3n^{3/2}}{2\sqrt{5+n^5}} \cdot \frac{1/n^{5/2}}{1/n^{5/2}}$$

$$= \lim_{n\to\infty} \frac{2+\frac{3}{n}}{2\sqrt{\frac{5}{n^5}+1}} = \frac{2+0}{2\sqrt{0+1}} = 1$$

Since $\sum b_n = 2 \cdot \sum \frac{1}{n^{1/2}}$ is divergent (p-series with $p = \frac{1}{2} \leq 1$)

the given series diverges by the Limit Comparison Test.