

11.3 The Integral Test and Estimates of Sums

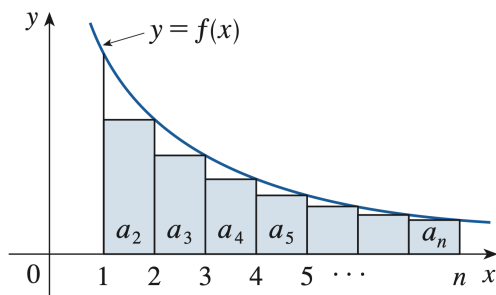
In general, it is difficult to find the exact sum of a series. We were able to accomplish this for geometric series and for some telescoping series because in each of those cases we could find a simple formula for the n th partial sum s_n . But usually it isn't easy to discover such a formula. Therefore, we develop several tests that enable us to determine whether a series is convergent or divergent without explicitly finding its sum. The first test involves improper integrals.

Theorem. Suppose $f(x)$ is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$.

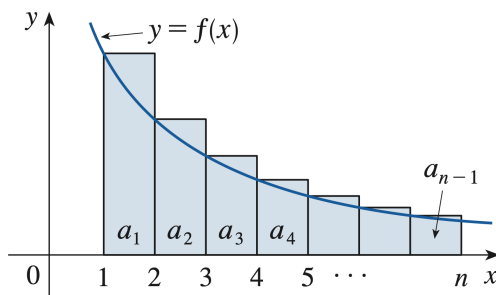
1. If $\int_1^{\infty} f(x) dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.
2. If $\int_1^{\infty} f(x) dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Proof.

- Show that $a_2 + a_3 + \cdots + a_n \leq \int_1^n f(x) dx$



- Show that $\int_1^n f(x) dx \leq a_1 + a_2 + \cdots + a_{n-1}$



- If $\int_1^\infty f(x) dx$ is convergent, show that $\sum a_n$ is convergent.

- If $\int_1^\infty f(x) dx$ is divergent, show that $\sum a_n$ is divergent.

□

Remark. When we use the Integral Test, it is not necessary to start the series or the integral at $n = 1$. For instance, in testing the series:

$$\sum_{n=4}^{\infty} \frac{1}{(n-3)^2},$$

we use:

$$\int_4^{\infty} \frac{1}{(x-3)^2} dx.$$

Also, it is not necessary that f be always decreasing. What is important is that f be *ultimately decreasing*, that is, decreasing for x larger than some number N . Then $\sum_{n=N}^{\infty} a_n$ is convergent, so $\sum_{n=1}^{\infty} a_n$ is convergent.

Example. Test the series:

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$$

for convergence or divergence.

Theorem. A p -series is a series of the form:

$$\sum_{n=1}^{\infty} \frac{1}{n^p},$$

where p is a real number. The behavior of the series depends on the value of p . In particular, it is convergent if $p > 1$ and divergent if $p \leq 1$.

Example. Determine if the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent or divergent.

Example. Determine if the series $\sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$ is convergent or divergent.

Example. Determine whether the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converges or diverges.

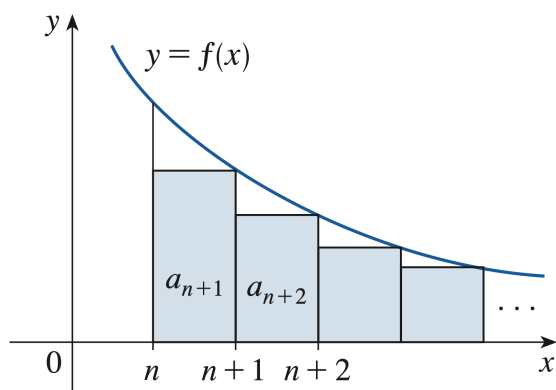
Estimating the Sum of a Series

Suppose we have been able to use the Integral Test to show that a series $\sum a_n$ is convergent and we now want to find an approximation to the sum s of the series. Any partial sum s_n is an approximation to s because $\lim_{n \rightarrow \infty} s_n = s$. But how good is such an approximation? To find out, we need to estimate the size of the *remainder*:

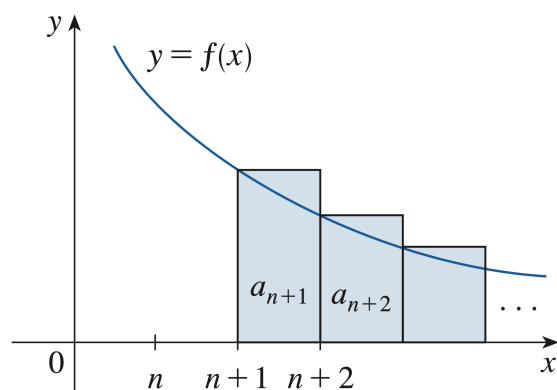
$$R_n = s - s_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots$$

Theorem. Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \geq n$, and $\sum a_n$ is convergent. If $R_n = s - s_n$, then:

$$\int_{n+1}^{\infty} f(x) dx \leq R_n \leq \int_n^{\infty} f(x) dx.$$



(a) Upper bound for R_n



(b) Lower bound for R_n

Example.

- (a) Approximate the sum of the series $\sum \frac{1}{n^3}$ by using the sum of the first 10 terms. Estimate the error involved in this approximation.
- (b) How many terms are required to ensure that the sum is accurate to within 0.0005?

Theorem. If we add s_n to each side of the inequalities for R_n , we obtain:

$$s_n + \int_{n+1}^{\infty} f(x) dx \leq s \leq s_n + \int_n^{\infty} f(x) dx.$$

These inequalities provide a lower bound and an upper bound for s , giving a more accurate approximation to the sum of the series than the partial sum s_n alone.

Example. Use the above theorem with $n = 10$ to estimate the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$.

Remark. If we compare this to the previous example, we see that the improved estimate for s can be much better than the estimate $s \approx s_n$. To make the error smaller than 0.0005, we had to use 32 terms in the previous example, but only 10 terms in the example above.