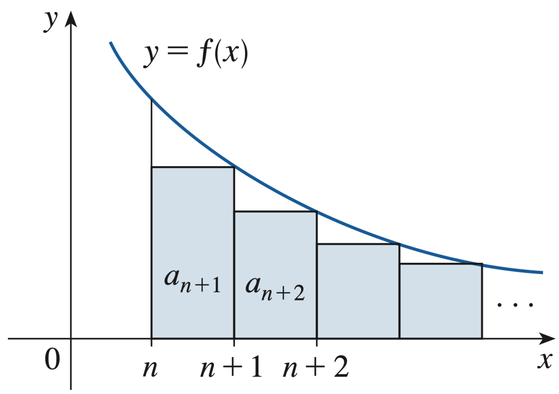
11.3 Integral Test Remainder Estimate

For alternating series, we developed a way to estimate the error made when approximating the sum of a convergent series by a partial sum. We now turn to the analogous question for series whose convergence is established by the Integral Test.

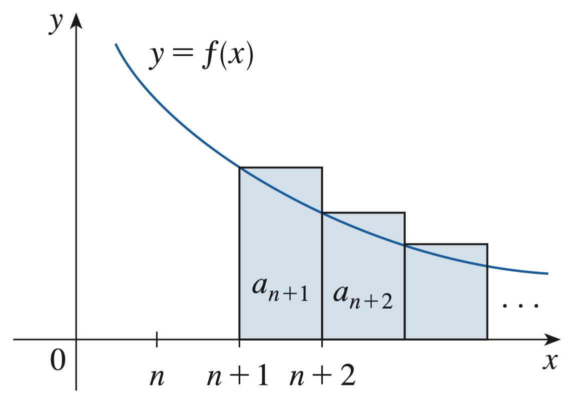
Theorem. Suppose $a_k = f(k)$, where f is continuous, positive, and decreasing for $x \geq n$. If the series $\sum_{k=1}^{\infty} a_k$ converges, and if $R_n = S - S_n$ denotes the remainder after n terms, then

$$\int_{n+1}^{\infty} f(x)\,dx \;\leq\; R_n \;\leq\; \int_{n}^{\infty} f(x)\,dx.$$

In other words, the error made by using the partial sum S_n is trapped between these two improper integrals.



(a) Upper bound for R_n



(b) Lower bound for R_n

Example. Suppose we use the sum of the first 10 terms to approximate the sum of the series

$$S = \sum_{n=1}^{\infty} \frac{1}{n^3}.$$

Use the Integral Test Remainder Estimate to find an upper bound for the error $R_{10} = S - S_{10}$.

Example. How many terms of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

are needed to guarantee that the partial sum is within 0.0005 of the true sum?