

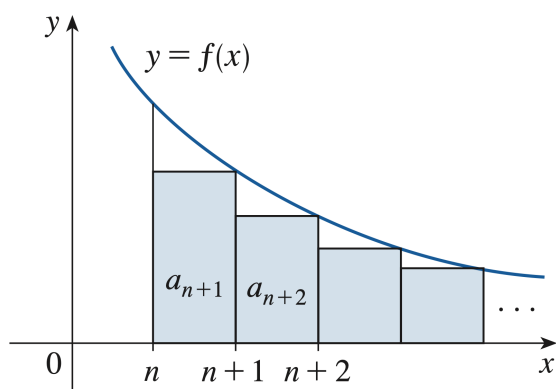
### 11.3 Integral Test Remainder Estimate

For alternating series, we developed a way to estimate the error made when approximating the sum of a convergent series by a partial sum. We now turn to the analogous question for series whose convergence is established by the Integral Test.

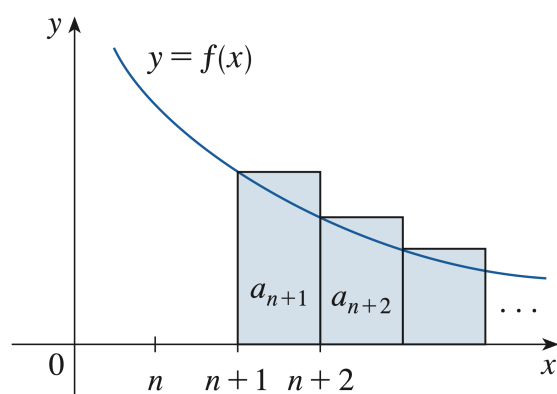
**Theorem.** Suppose  $a_k = f(k)$ , where  $f$  is continuous, positive, and decreasing for  $x \geq n$ . If the series  $\sum_{k=1}^{\infty} a_k$  converges, and if  $R_n = S - S_n$  denotes the remainder after  $n$  terms, then

$$\int_{n+1}^{\infty} f(x) dx \leq R_n \leq \int_n^{\infty} f(x) dx.$$

In other words, the error made by using the partial sum  $S_n$  is trapped between these two improper integrals.



(a) Upper bound for  $R_n$



(b) Lower bound for  $R_n$

$$R_n = S - S_n = \sum_{k=1}^{\infty} a_k - \sum_{k=1}^n a_k = a_{n+1} + a_{n+2} + a_{n+3} + \dots$$

From (a),  $R_n \leq \int_n^{\infty} f(x) dx$  (RRAM)

From (b),  $R_n \geq \int_{n+1}^{\infty} f(x) dx$  (LRAM)

**Example.** Suppose we use the sum of the first 10 terms to approximate the sum of the series

$$S = \sum_{n=1}^{\infty} \frac{1}{n^3}.$$

Use the Integral Test Remainder Estimate to find an upper bound for the error  $R_{10} = S - S_{10}$ .

$$\begin{aligned} R_{10} &\leq \int_{10}^{\infty} \frac{1}{x^3} dx = \lim_{t \rightarrow \infty} \int_{10}^t \frac{1}{x^3} dx = \lim_{t \rightarrow \infty} \left[ -\frac{1}{2x^2} \right]_{10}^t \\ &= \lim_{t \rightarrow \infty} \left( -\frac{1}{2t^2} + \frac{1}{2 \cdot 10^2} \right) \\ &= \frac{1}{200} = 0.005 \end{aligned}$$

$$S_{10} = \frac{1}{1^3} + \frac{1}{2^3} + \dots + \frac{1}{10^3} \approx 1.1975 \text{ is within } 0.005 \text{ of } \sum_{n=1}^{\infty} \frac{1}{n^3}$$

**Example.** How many terms of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

are needed to guarantee that the partial sum is within 0.0005 of the true sum?

• In general,  $R_n \leq \int_n^{\infty} \frac{1}{x^3} dx = \frac{1}{2 \cdot n^2}$  [see above]

• Solve for  $n$  so that  $\frac{1}{2 \cdot n^2} \leq 0.0005$

$$\Rightarrow \frac{1}{n^2} \leq 0.001$$

$$\Rightarrow n^2 \geq \frac{1}{0.001} = 1000$$

$$\Rightarrow n \geq \sqrt{1000} \approx 31.6$$

• We need  $n = 32$  terms to be within 0.0005