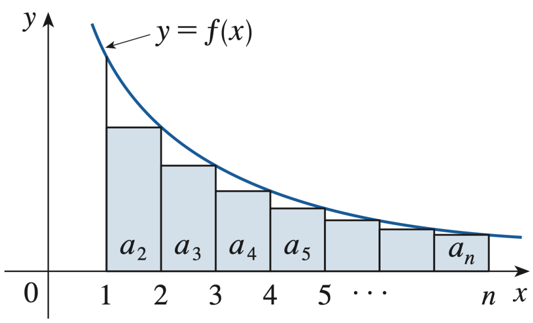
11.3 The Integral Test

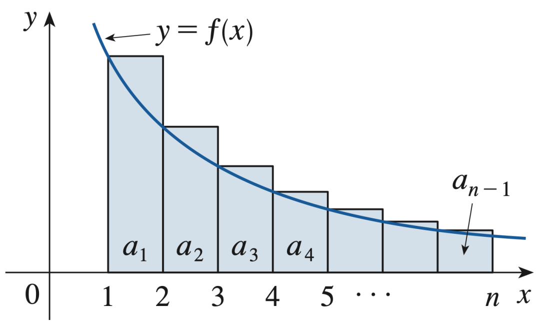
In general, it is difficult to find the exact sum of a series. We were able to accomplish this for geometric series and for some telescoping series because in each of those cases we could find a simple formula for the nth partial sum s_n. But usually it isn't easy to discover such a formula. Therefore, we develop several tests that enable us to determine whether a series is convergent or divergent without explicitly finding its sum. The first test involves improper integrals.

Theorem. Suppose $f(x)$ is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$.

1. If $\displaystyle\int_1^\infty f(x)\,dx$ is convergent, then $\displaystyle\sum_{n=1}^\infty a_n$ is convergent.
2. If $\displaystyle\int_1^\infty f(x)\,dx$ is divergent, then $\displaystyle\sum_{n=1}^\infty a_n$ is divergent.

Proof.





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Example. Determine whether the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$$

converges or diverges.

Remark. When we use the Integral Test, it is not necessary to start the series or the integral at $n = 1$. For instance, in testing the series:

$$\sum_{n=4}^{\infty} \frac{1}{(n-3)^2},$$

we use:

$$\int_4^{\infty} \frac{1}{(x-3)^2}\, dx.$$

Also, it is not necessary that f be always decreasing. What is important is that f be *ultimately decreasing*, that is, decreasing for x larger than some number N. Then $\sum_{n=N}^{\infty} a_n$ is convergent, so $\sum_{n=1}^{\infty} a_n$ is convergent.

Example. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converges or diverges.

Theorem. A p**-series** is a series of the form:

$$\sum_{n=1}^{\infty} \frac{1}{n^p},$$

where p is a real number. The behavior of the series depends on the value of p. In particular, it is convergent if $p > 1$ and divergent if $p \leq 1$.

Example. Determine if the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent or divergent.

Example. Determine if the series $\sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$ is convergent or divergent.