

## Integral Test

1.  $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

Let

$$f(x) = \frac{1}{x \ln x}.$$

For  $x \geq 2$ ,  $f$  is continuous and positive. Also,

$$f'(x) = -\frac{\ln x + 1}{x^2(\ln x)^2}.$$

Since  $x \geq 2$  implies  $\ln x > 0$ , we have  $\ln x + 1 > 0$ , so

$$f'(x) < 0 \quad \text{for } x \geq 2.$$

Thus  $f$  is decreasing on  $[2, \infty)$ , so the Integral Test applies.

Now compute:

$$\int_2^{\infty} \frac{1}{x \ln x} dx = \lim_{t \rightarrow \infty} \int_2^t \frac{1}{x \ln x} dx.$$

Let  $u = \ln x$ , so  $du = \frac{1}{x} dx$ . Then

$$\int_2^t \frac{1}{x \ln x} dx = \int_{\ln 2}^{\ln t} \frac{1}{u} du = \ln |u| \Big|_{\ln 2}^{\ln t} = \ln(\ln t) - \ln(\ln 2).$$

As  $t \rightarrow \infty$ , we have  $\ln t \rightarrow \infty$ , so

$$\ln(\ln t) - \ln(\ln 2) \rightarrow \infty.$$

Therefore,

$$\int_2^{\infty} \frac{1}{x \ln x} dx$$

diverges, so

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

diverges by the Integral Test.

2.  $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

Let

$$f(x) = \frac{1}{x(\ln x)^2}.$$

For  $x \geq 2$ ,  $f$  is continuous and positive. Also,

$$f'(x) = -\frac{\ln x + 2}{x^2(\ln x)^3}.$$

Since  $x \geq 2$  implies  $\ln x > 0$ , we have  $\ln x + 2 > 0$ , so

$$f'(x) < 0 \quad \text{for } x \geq 2.$$

Thus  $f$  is decreasing on  $[2, \infty)$ , so the Integral Test applies.

Now compute:

$$\int_2^{\infty} \frac{1}{x(\ln x)^2} dx = \lim_{t \rightarrow \infty} \int_2^t \frac{1}{x(\ln x)^2} dx.$$

Let  $u = \ln x$ , so  $du = \frac{1}{x} dx$ . Then

$$\int_2^t \frac{1}{x(\ln x)^2} dx = \int_{\ln 2}^{\ln t} u^{-2} du = \left[ -\frac{1}{u} \right]_{\ln 2}^{\ln t} = -\frac{1}{\ln t} + \frac{1}{\ln 2}.$$

As  $t \rightarrow \infty$ , we have  $\frac{1}{\ln t} \rightarrow 0$ , so

$$\int_2^{\infty} \frac{1}{x(\ln x)^2} dx = \frac{1}{\ln 2}.$$

Therefore,

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$$

converges by the Integral Test.

3. 
$$\sum_{n=3}^{\infty} \frac{1}{n(\ln n)(\ln \ln n)}$$

Let

$$f(x) = \frac{1}{x(\ln x)(\ln \ln x)}.$$

For  $x \geq 3$ ,  $f$  is continuous and positive. To verify that  $f$  is decreasing, write

$$g(x) = x(\ln x)(\ln \ln x), \quad \text{so} \quad f(x) = \frac{1}{g(x)}.$$

Then

$$g'(x) = (\ln x)(\ln \ln x) + (\ln \ln x) + 1.$$

For  $x \geq 3$ , we have  $\ln x > 0$  and  $\ln \ln x > 0$ , so every term on the right is positive. Hence

$$g'(x) > 0 \quad \text{for } x \geq 3.$$

Thus  $g$  is increasing, which means  $f(x) = 1/g(x)$  is decreasing on  $[3, \infty)$ . So the Integral Test applies.

Now compute:

$$\int_3^{\infty} \frac{1}{x(\ln x)(\ln \ln x)} dx = \lim_{t \rightarrow \infty} \int_3^t \frac{1}{x(\ln x)(\ln \ln x)} dx.$$

Let

$$u = \ln \ln x.$$

Then

$$du = \frac{1}{x \ln x} dx.$$

So

$$\int_3^t \frac{1}{x(\ln x)(\ln \ln x)} dx = \int_{\ln \ln 3}^{\ln \ln t} \frac{1}{u} du = \ln |u| \Big|_{\ln \ln 3}^{\ln \ln t} = \ln(\ln \ln t) - \ln(\ln \ln 3).$$

As  $t \rightarrow \infty$ , we have  $\ln \ln t \rightarrow \infty$ , so

$$\ln(\ln \ln t) - \ln(\ln \ln 3) \rightarrow \infty.$$

Therefore,

$$\int_3^{\infty} \frac{1}{x(\ln x)(\ln \ln x)} dx$$

diverges, so

$$\sum_{n=3}^{\infty} \frac{1}{n(\ln n)(\ln \ln n)}$$

diverges by the Integral Test.

$$4. \sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$$

Let

$$f(x) = \frac{1}{x\sqrt{\ln x}}.$$

For  $x \geq 2$ ,  $f$  is continuous and positive. Also,

$$f'(x) = -\frac{2\ln x + 1}{2x^2(\ln x)^{3/2}}.$$

Since  $x \geq 2$  implies  $\ln x > 0$ , we have  $2\ln x + 1 > 0$ , so

$$f'(x) < 0 \quad \text{for } x \geq 2.$$

Thus  $f$  is decreasing on  $[2, \infty)$ , so the Integral Test applies.

Now compute:

$$\int_2^{\infty} \frac{1}{x\sqrt{\ln x}} dx = \lim_{t \rightarrow \infty} \int_2^t \frac{1}{x\sqrt{\ln x}} dx.$$

Let

$$u = \sqrt{\ln x}.$$

Then

$$du = \frac{1}{2x\sqrt{\ln x}} dx, \quad \text{so} \quad \frac{1}{x\sqrt{\ln x}} dx = 2 du.$$

Therefore,

$$\int_2^t \frac{1}{x\sqrt{\ln x}} dx = \int_{\sqrt{\ln 2}}^{\sqrt{\ln t}} 2 du = 2u \Big|_{\sqrt{\ln 2}}^{\sqrt{\ln t}} = 2\sqrt{\ln t} - 2\sqrt{\ln 2}.$$

As  $t \rightarrow \infty$ , we have  $\sqrt{\ln t} \rightarrow \infty$ , so

$$2\sqrt{\ln t} - 2\sqrt{\ln 2} \rightarrow \infty.$$

Therefore,

$$\int_2^{\infty} \frac{1}{x\sqrt{\ln x}} dx$$

diverges, so

$$\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$$

diverges by the Integral Test.

$$5. \sum_{n=1}^{\infty} \frac{1}{n^p} \quad (\text{general case})$$

Let

$$f(x) = \frac{1}{x^p} = x^{-p}.$$

Assume  $p > 0$ . Then for  $x \geq 1$ ,  $f$  is continuous and positive. Also,

$$f'(x) = -px^{-p-1}.$$

Since  $p > 0$  and  $x \geq 1$ , we have

$$f'(x) < 0 \quad \text{for } x \geq 1.$$

Thus  $f$  is decreasing on  $[1, \infty)$ , so the Integral Test applies.

We now compute

$$\int_1^{\infty} \frac{1}{x^p} dx = \lim_{t \rightarrow \infty} \int_1^t x^{-p} dx.$$

**Case 1:**  $p \neq 1$ .

$$\int_1^t x^{-p} dx = \left[ \frac{x^{1-p}}{1-p} \right]_1^t = \frac{t^{1-p} - 1}{1-p}.$$

If  $p > 1$ , then  $1 - p < 0$ , so  $t^{1-p} \rightarrow 0$ . Hence

$$\lim_{t \rightarrow \infty} \frac{t^{1-p} - 1}{1-p} = \frac{0 - 1}{1-p} = \frac{1}{p-1}.$$

So the integral converges.

If  $0 < p < 1$ , then  $1 - p > 0$ , so  $t^{1-p} \rightarrow \infty$ . Hence

$$\frac{t^{1-p} - 1}{1-p} \rightarrow \infty,$$

so the integral diverges.

**Case 2:**  $p = 1$ .

$$\int_1^t \frac{1}{x} dx = \ln x \Big|_1^t = \ln t.$$

Since  $\ln t \rightarrow \infty$ , the integral diverges.

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges if and only if

$$p > 1.$$

6.  $\sum_{n=2}^{\infty} \frac{\sqrt{\ln n}}{n}$

Let

$$f(x) = \frac{\sqrt{\ln x}}{x}.$$

For  $x \geq 2$ ,  $f$  is continuous and positive. Also,

$$f'(x) = \frac{1 - 2 \ln x}{2x^2 \sqrt{\ln x}}.$$

Since  $x \geq 2$  implies  $\ln 2 > \frac{1}{2}$ , we have

$$\ln x \geq \ln 2 > \frac{1}{2},$$

so

$$1 - 2 \ln x < 0.$$

Therefore,

$$f'(x) < 0 \quad \text{for } x \geq 2.$$

Thus  $f$  is decreasing on  $[2, \infty)$ , so the Integral Test applies.

Now compute:

$$\int_2^{\infty} \frac{\sqrt{\ln x}}{x} dx = \lim_{t \rightarrow \infty} \int_2^t \frac{\sqrt{\ln x}}{x} dx.$$

Let

$$u = \ln x, \quad du = \frac{1}{x} dx.$$

Then

$$\int_2^t \frac{\sqrt{\ln x}}{x} dx = \int_{\ln 2}^{\ln t} \sqrt{u} du = \int_{\ln 2}^{\ln t} u^{1/2} du = \frac{2}{3} u^{3/2} \Big|_{\ln 2}^{\ln t}.$$

So

$$\int_2^t \frac{\sqrt{\ln x}}{x} dx = \frac{2}{3} (\ln t)^{3/2} - \frac{2}{3} (\ln 2)^{3/2}.$$

As  $t \rightarrow \infty$ , this tends to  $\infty$ . Therefore,

$$\int_2^{\infty} \frac{\sqrt{\ln x}}{x} dx$$

diverges, so

$$\sum_{n=2}^{\infty} \frac{\sqrt{\ln n}}{n}$$

diverges by the Integral Test.

7.  $\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$

Let

$$f(x) = \frac{1}{x^2 + 1}.$$

For  $x \geq 1$ ,  $f$  is continuous and positive. Also,

$$f'(x) = -\frac{2x}{(x^2 + 1)^2}.$$

Since  $x \geq 1$ , we have  $f'(x) < 0$ . Thus  $f$  is decreasing on  $[1, \infty)$ , so the Integral Test applies.

Now compute:

$$\int_1^{\infty} \frac{1}{x^2 + 1} dx = \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x^2 + 1} dx = \lim_{t \rightarrow \infty} \tan^{-1}(x) \Big|_1^t.$$

Thus

$$\int_1^{\infty} \frac{1}{x^2 + 1} dx = \lim_{t \rightarrow \infty} \left( \tan^{-1}(t) - \frac{\pi}{4} \right) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$$

converges by the Integral Test.

8.  $\sum_{n=1}^{\infty} \frac{2n}{n^4 + 25}$

Let

$$f(x) = \frac{2x}{x^4 + 25}.$$

For  $x \geq 1$ ,  $f$  is continuous and positive. Also,

$$f'(x) = \frac{2(x^4 + 25) - 8x^4}{(x^4 + 25)^2} = \frac{50 - 6x^4}{(x^4 + 25)^2}.$$

If  $x \geq 2$ , then  $x^4 \geq 16$ , so

$$50 - 6x^4 \leq 50 - 96 < 0.$$

Thus

$$f'(x) < 0 \quad \text{for } x \geq 2.$$

So  $f$  is eventually decreasing, and the Integral Test applies.

Now compute:

$$\int_2^\infty \frac{2x}{x^4 + 25} dx = \lim_{t \rightarrow \infty} \int_2^t \frac{2x}{x^4 + 25} dx.$$

Let

$$u = x^2, \quad du = 2x dx.$$

Then

$$\int_2^t \frac{2x}{x^4 + 25} dx = \int_4^{t^2} \frac{1}{u^2 + 25} du = \frac{1}{5} \tan^{-1}\left(\frac{u}{5}\right) \Big|_4^{t^2}.$$

So

$$\int_2^t \frac{2x}{x^4 + 25} dx = \frac{1}{5} \tan^{-1}\left(\frac{t^2}{5}\right) - \frac{1}{5} \tan^{-1}\left(\frac{4}{5}\right).$$

As  $t \rightarrow \infty$ ,

$$\tan^{-1}\left(\frac{t^2}{5}\right) \rightarrow \frac{\pi}{2}.$$

Hence

$$\int_2^\infty \frac{2x}{x^4 + 25} dx = \frac{1}{5} \left( \frac{\pi}{2} - \tan^{-1}\left(\frac{4}{5}\right) \right).$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{2n}{n^4 + 25}$$

converges by the Integral Test.

9. 
$$\sum_{n=1}^{\infty} \frac{3n^2}{n^6 + 36}$$

Let

$$f(x) = \frac{3x^2}{x^6 + 36}.$$

For  $x \geq 1$ ,  $f$  is continuous and positive. Also,

$$f'(x) = \frac{6x(x^6 + 36) - 18x^7}{(x^6 + 36)^2} = \frac{12x(18 - x^6)}{(x^6 + 36)^2}.$$

If  $x \geq 2$ , then  $x^6 \geq 64$ , so

$$18 - x^6 < 0.$$

Thus

$$f'(x) < 0 \quad \text{for } x \geq 2.$$

So  $f$  is eventually decreasing, and the Integral Test applies.

Now compute:

$$\int_2^\infty \frac{3x^2}{x^6 + 36} dx = \lim_{t \rightarrow \infty} \int_2^t \frac{3x^2}{x^6 + 36} dx.$$

Let

$$u = x^3, \quad du = 3x^2 dx.$$

Then

$$\int_2^t \frac{3x^2}{x^6 + 36} dx = \int_8^{t^3} \frac{1}{u^2 + 36} du = \frac{1}{6} \tan^{-1}\left(\frac{u}{6}\right) \Big|_8^{t^3}.$$

So

$$\int_2^t \frac{3x^2}{x^6 + 36} dx = \frac{1}{6} \tan^{-1}\left(\frac{t^3}{6}\right) - \frac{1}{6} \tan^{-1}\left(\frac{8}{6}\right).$$

As  $t \rightarrow \infty$ ,

$$\tan^{-1}\left(\frac{t^3}{6}\right) \rightarrow \frac{\pi}{2}.$$

Hence

$$\int_2^\infty \frac{3x^2}{x^6 + 36} dx = \frac{1}{6} \left( \frac{\pi}{2} - \tan^{-1}\left(\frac{4}{3}\right) \right).$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{3n^2}{n^6 + 36}$$

converges by the Integral Test.

10.  $\sum_{n=1}^{\infty} \frac{4n^3}{n^8 + 64}$

Let

$$f(x) = \frac{4x^3}{x^8 + 64}.$$

For  $x \geq 1$ ,  $f$  is continuous and positive. Also,

$$f'(x) = \frac{12x^2(x^8 + 64) - 32x^{10}}{(x^8 + 64)^2} = \frac{4x^2(192 - 5x^8)}{(x^8 + 64)^2}.$$

If  $x \geq 2$ , then  $x^8 \geq 256$ , so

$$192 - 5x^8 \leq 192 - 1280 < 0.$$

Thus

$$f'(x) < 0 \quad \text{for } x \geq 2.$$

So  $f$  is eventually decreasing, and the Integral Test applies.

Now compute:

$$\int_2^\infty \frac{4x^3}{x^8 + 64} dx = \lim_{t \rightarrow \infty} \int_2^t \frac{4x^3}{x^8 + 64} dx.$$

Let

$$u = x^4, \quad du = 4x^3 dx.$$

Then

$$\int_2^t \frac{4x^3}{x^8 + 64} dx = \int_{16}^{t^4} \frac{1}{u^2 + 64} du = \frac{1}{8} \tan^{-1}\left(\frac{u}{8}\right) \Big|_{16}^{t^4}.$$

So

$$\int_2^t \frac{4x^3}{x^8 + 64} dx = \frac{1}{8} \tan^{-1}\left(\frac{t^4}{8}\right) - \frac{1}{8} \tan^{-1}(2).$$

As  $t \rightarrow \infty$ ,

$$\tan^{-1}\left(\frac{t^4}{8}\right) \rightarrow \frac{\pi}{2}.$$

Hence

$$\int_2^\infty \frac{4x^3}{x^8 + 64} dx = \frac{1}{8} \left( \frac{\pi}{2} - \tan^{-1}(2) \right).$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{4n^3}{n^8 + 64}$$

converges by the Integral Test.