

11.3 The Integral Test

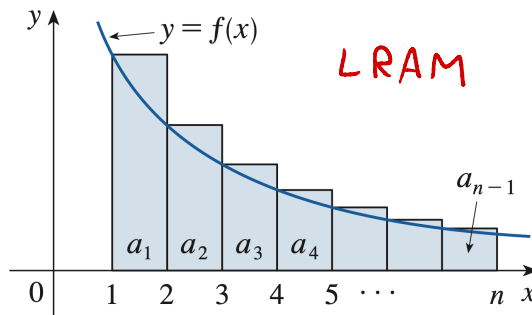
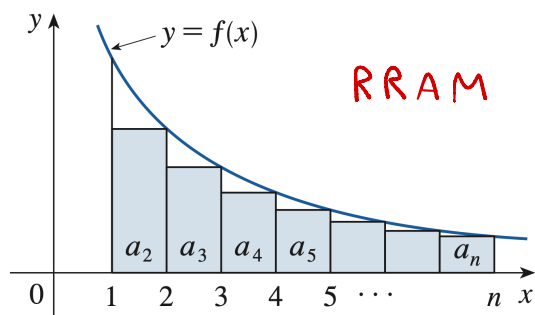
In general, it is difficult to find the exact sum of a series. We were able to accomplish this for geometric series and for some telescoping series because in each of those cases we could find a simple formula for the n th partial sum s_n . But usually it isn't easy to discover such a formula. Therefore, we develop several tests that enable us to determine whether a series is convergent or divergent without explicitly finding its sum. The first test involves improper integrals.

Theorem. Suppose $f(x)$ is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$.

1. If $\int_1^{\infty} f(x) dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

2. If $\int_1^{\infty} f(x) dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Proof.



key observation: $0 \leq \sum_{k=2}^n a_k \leq \int_1^n f(x) dx \leq \sum_{k=1}^{n-1} a_k$

① If $\lim_{n \rightarrow \infty} \int_1^n f(x) dx$ is finite, then $\sum_{k=2}^{\infty} a_k$ converges

so $a_1 + \sum_{k=2}^{\infty} a_k = \sum_{k=1}^{\infty} a_k$ converges

② If $\lim_{n \rightarrow \infty} \int_1^n f(x) dx$ is infinite, then $\lim_{n \rightarrow \infty} \sum_{k=1}^{n-1} a_k$ diverges.

so $\sum_{k=1}^{\infty} a_k$ diverges.

□

Example. Determine whether the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$$

converges or diverges.

Need a formal introduction on exams where you state the test and check all conditions.

Introduction: The function $f(x) = \frac{1}{x^2+1}$ is continuous and positive on $[1, \infty)$. To show $f(x)$ is decreasing, compute $f'(x)$:

$$f'(x) = -(x^2+1)^{-2} \cdot 2x = \frac{-2x}{(x^2+1)^2}$$

$f'(x) < 0$ for $x > 1$. The Integral Test applies.

Apply the Test:

$$\begin{aligned} \int_1^{\infty} \frac{1}{x^2+1} dx &= \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x^2+1} dx = \lim_{t \rightarrow \infty} \left[\tan^{-1} x \right]_1^t \\ &= \lim_{t \rightarrow \infty} \left(\tan^{-1}(t) - \tan^{-1}(1) \right) = \lim_{t \rightarrow \infty} \tan^{-1}(t) - \lim_{t \rightarrow \infty} \tan^{-1}(1) \\ &= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} \end{aligned}$$

Conclusion: Since $\int_1^{\infty} \frac{1}{x^2+1} dx$ converges, the series

$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges by the Integral Test.

Need a formal conclusion on exams:

"Because _____, the series converges/diverges by the _____ test."

Remark. When we use the Integral Test, it is not necessary to start the series or the integral at $n = 1$. For instance, in testing the series:

$$\sum_{n=4}^{\infty} \frac{1}{(n-3)^2},$$

we use:

$$\int_4^{\infty} \frac{1}{(x-3)^2} dx.$$

Also, it is not necessary that f be always decreasing. What is important is that f be *ultimately decreasing*, that is, decreasing for x larger than some number N . Then $\sum_{n=N}^{\infty} a_n$ is convergent, so $\sum_{n=1}^{\infty} a_n$ is convergent.

Example. Determine whether the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converges or diverges.

Introduction: The function $f(x) = \frac{\ln x}{x}$ is positive and continuous for $x > 1$. To show $f(x)$ is decreasing, compute $f'(x)$:

$$f'(x) = \frac{x \cdot \frac{1}{x} - \ln(x)}{x^2} = \frac{1 - \ln(x)}{x^2}$$

$f'(x) < 0$ when $\ln(x) > 1$, that is, when $x > e$. The Integral Test applies.

Apply the Test:

$$\begin{aligned} \int_1^{\infty} \frac{\ln x}{x} dx &= \lim_{t \rightarrow \infty} \int_1^t \frac{\ln x}{x} dx = \lim_{t \rightarrow \infty} \left[\frac{(\ln(x))^2}{2} \right]_1^t \\ &= \lim_{t \rightarrow \infty} \left(\frac{(\ln(t))^2}{2} - \frac{(\ln(1))^2}{2} \right) = \lim_{t \rightarrow \infty} \frac{(\ln(t))^2}{2} = \infty \end{aligned}$$

Conclusion: Since $\int_1^{\infty} \frac{\ln x}{x} dx$ diverges, the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ diverges by the Integral Test.

Theorem. A p -series is a series of the form:

$$\sum_{n=1}^{\infty} \frac{1}{n^p},$$

where p is a real number. The behavior of the series depends on the value of p . In particular, it is convergent if $p > 1$ and divergent if $p \leq 1$.

• If $p \leq 0$: $\lim_{n \rightarrow \infty} \frac{1}{n^p} \neq 0$. The series diverges by the Test for Divergence.

• If $p > 0$: The function $f(x) = \frac{1}{x^p}$ is continuous, positive, and decreasing on $[1, \infty)$. The Integral Test applies.

In §7.8, we showed $\int_1^{\infty} \frac{1}{x^p} dx$ converges if $p > 1$ and diverges if $0 < p \leq 1$.

Example. Determine if the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent or divergent.

This is a p -series with $p=3$. Since $p=3 > 1$, the series converges.

Example. Determine if the series $\sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$ is convergent or divergent.

This is a p -series with $p=\frac{1}{3}$. Since $p=\frac{1}{3} \leq 1$, the series diverges.