11.3 The Integral Test and Estimates of Sums

In general, it is difficult to find the exact sum of a series. We were able to accomplish this for geometric series and for some telescoping series because in each of those cases we could find a simple formula for the nth partial sum s_n. But usually it isn't easy to discover such a formula. Therefore, we develop several tests that enable us to determine whether a series is convergent or divergent without explicitly finding its sum. The first test involves improper integrals.

Theorem. Suppose $f(x)$ is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$.

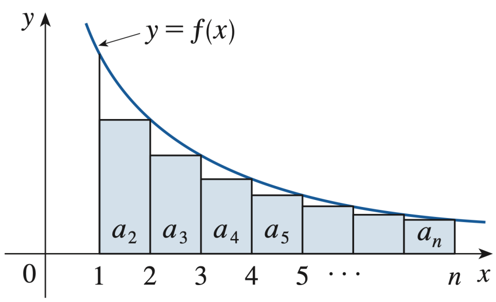
1. If $\displaystyle\int_1^\infty f(x)\,dx$ is convergent, then $\displaystyle\sum_{n=1}^\infty a_n$ is convergent.

2. If $\displaystyle\int_1^\infty f(x)\,dx$ is divergent, then $\displaystyle\sum_{n=1}^\infty a_n$ is divergent.

Proof.

- Show that $a_2 + a_3 + \cdots + a_n \le \int_1^n f(x)\,dx$

RRAM



$a_2 + a_3 + \cdots + a_n$ represents the area of the shaded rectangles, which is less than or equal to the total area under the curve.

- Show that $\int_1^n f(x)\,dx \le a_1 + a_2 + \cdots + a_{n-1}$

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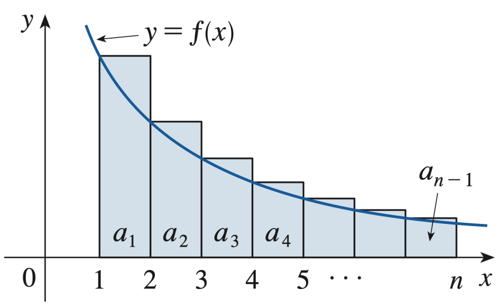


$a_1 + a_2 + \cdots + a_{n-1}$ represents the area of the shaded rectangles, which is greater than or equal to the total area under the curve.

- If $\int_1^\infty f(x)\,dx$ is convergent, show that $\sum a_n$ is convergent.

We will show $\{s_n\}$ is an increasing, bounded sequence. This will imply that $\{s_n\}$ converges by the Monotonic Sequence theorem. By definition, this will show $\sum a_n$ converges.

Increasing: we need to show $s_n < s_{n+1}$. Since $a_{n+1} = f(n+1)$ and $f(x)$ is a positive function, $s_n < s_n + a_{n+1} = s_{n+1}$

Bounded: $a_2 + a_3 + \cdots + a_n \leq \int_1^n f(x)\,dx \leq \int_1^\infty f(x)\,dx$

added a_1 to both sides

So... $s_n \leq a_1 + \int_1^\infty f(x)\,dx = M$, where M is some finite constant.

Conclude: $s_n \leq M$ for all n. Hence $\{s_n\}$ is bounded above.

- If $\int_1^\infty f(x)\,dx$ is divergent, show that $\sum a_n$ is divergent.

Recall: $\int_1^n f(x)\,dx \leq a_1 + a_2 + \cdots + a_{n-1} = s_{n-1}$

Since $\int_1^n f(x)\,dx \to \infty$ as $n \to \infty$, $s_{n-1} \to \infty$ as well.

Since $s_{n-1} \leq s_n$ this implies $s_n \to \infty$, and so $\sum a_n$ diverges.

$\square$

Remark. When we use the Integral Test, it is not necessary to start the series or the integral at $n = 1$. For instance, in testing the series:

$$\sum_{n=4}^{\infty} \frac{1}{(n-3)^2},$$

we use:

$$\int_4^{\infty} \frac{1}{(x-3)^2}\, dx.$$

Also, it is not necessary that f be always decreasing. What is important is that f be *ultimately decreasing*, that is, decreasing for x larger than some number N. Then $\sum_{n=N}^{\infty} a_n$ is convergent, so $\sum_{n=1}^{\infty} a_n$ is convergent.

Example. Test the series:

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$$

for convergence or divergence.

The function $f(x) = \frac{1}{x^2+1}$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies.

$$\int_1^{\infty} \frac{1}{x^2+1}\, dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2+1}\, dx$$

$$= \lim_{t\to\infty} \left[\tan^{-1} x\right]_1^t$$

$$= \lim_{t\to\infty} \left(\tan^{-1}(t) - \tan^{-1}(1)\right)$$

$$= \lim_{t\to\infty} \tan^{-1} t - \lim_{t\to\infty} \tan^{-1} 1$$

$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

Thus, $\int_1^{\infty} \frac{1}{x^2+1}\, dx$ converges. So, by the Integral Test, the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges as well.

Theorem. A p**-series** is a series of the form:

$$\sum_{n=1}^{\infty} \frac{1}{n^p},$$

where p is a real number. The behavior of the series depends on the value of p. In particular, it is convergent if $p > 1$ and divergent if $p \leq 1$.

- If $p < 0$: $\lim_{n\to\infty} \frac{1}{n^p} = \infty$. By the Test for Divergence, the series diverges.

- If $p = 0$: $\lim_{n\to\infty} \frac{1}{n^p} = \lim_{n\to\infty} \frac{1}{1} = 1$. By the Test for Divergence, the series diverges.

- If $p > 0$: The function $f(x) = \frac{1}{x^p}$ is continuous, positive, and decreasing on $[1, \infty)$. The Integral Test applies. In §7.8, we showed $\int_1^\infty \frac{1}{x^p}\,dx$ converges if $p > 1$ and diverges if $0 < p \leq 1$. The same can be said for the series by the Integral Test.

Example. Determine if the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent or divergent.

This is a p-series with $p=3$. Since $3>1$, the series converges.

Example. Determine if the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$ is convergent or divergent.

This is a p-series with $p=\frac{1}{3}$. Since $\frac{1}{3} \le 1$, the series diverges.

Example. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converges or diverges.

WebAssign

not a p-series

The function $f(x) = \frac{\ln x}{x}$ is positive and continuous for $x>1$.

To show $f(x)$ is decreasing, we compute the derivative:

$$f'(x) = \frac{x \cdot \frac{1}{x} - \ln x}{x^2} = \frac{1-\ln x}{x^2}$$

$f'(x) < 0$ when $\ln x > 1$, that is, when $x > e$. So f is decreasing when $x > e$.

Apply the Integral Test:

$$\int_1^\infty \frac{\ln x}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{\ln x}{x}\,dx = \lim_{t\to\infty} \left[\frac{(\ln x)^2}{2}\right]_1^t$$

$$= \lim_{t\to\infty} \frac{(\ln t)^2}{2} - \frac{(\ln(1))^2}{2} = \lim_{t\to\infty} \frac{(\ln t)^2}{2} \to \infty$$

Since the integral diverges, the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$ diverges by the Integral Test.

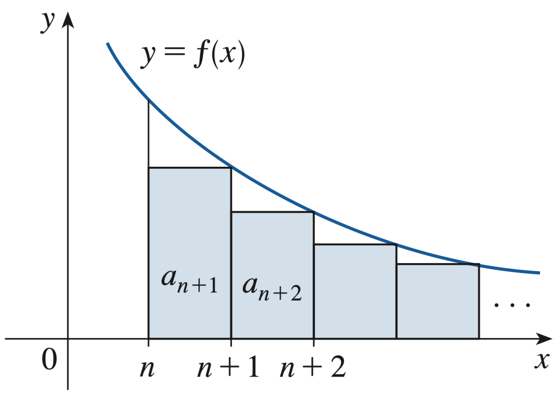
Estimating the Sum of a Series

Suppose we have been able to use the Integral Test to show that a series $\sum a_n$ is convergent and we now want to find an approximation to the sum s of the series. Any partial sum s_n is an approximation to s because $\lim_{n\to\infty} s_n = s$. But how good is such an approximation? To find out, we need to estimate the size of the *remainder*:

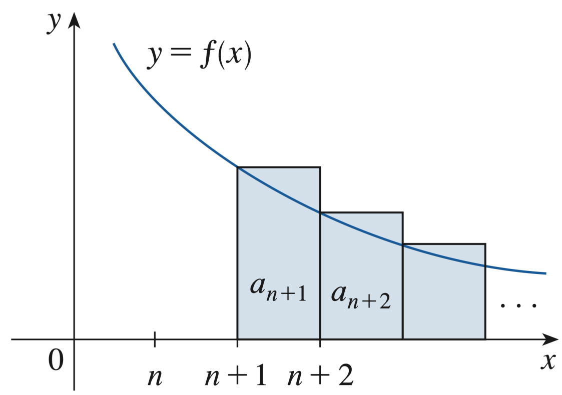
$$R_n = s - s_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots$$

Theorem. Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \geq n$, and $\sum a_n$ is convergent. If $R_n = s - s_n$, then:

$$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx.$$



(a) Upper bound for R_n



(b) Lower bound for R_n

$S - S_n$ = Actual - Estimate = R_n = $a_{n+1} + a_{n+2} + a_{n+3} + \cdots$ = sum of the rectangles

For $x \geq n$, compare the areas of the rectangles to the area under the curve

From (a), $R_n \leq \int_n^{\infty} f(x)\,dx$

From (b), $R_n \geq \int_{n+1}^{\infty} f(x)\,dx$

Example.

(a) Approximate the sum of the series $\sum \frac{1}{n^3}$ by using the sum of the first 10 terms. Estimate the error involved in this approximation.

(b) How many terms are required to ensure that the sum is accurate to within 0.0005?

Note: $f(x) = \frac{1}{x^3}$ satisfies the conditions for the Integral Test (positive, continuous, decreasing for $x \geq 1$)

(a) $$\sum_{n=1}^{\infty} \frac{1}{n^3} \approx S_{10} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots + \frac{1}{10^3} \approx 1.1975$$

According to the remainder estimate,

$$R_n \leq \int_n^{\infty} \frac{1}{x^3}\,dx = \lim_{t\to\infty} \int_n^t \frac{1}{x^3}\,dx = \lim_{t\to\infty} \left[-\frac{1}{2x^2} \right]_n^t$$

$$= \lim_{t\to\infty} \left(-\frac{1}{2t^2} + \frac{1}{2n^2} \right) = \frac{1}{2n^2}$$

$$\Rightarrow R_{10} \leq \frac{1}{2(10)^2} = \frac{1}{200} = 0.005$$

Hence the estimate $S_{10} \approx 1.1975$ is at most 0.005 away from the actual sum.

(b) By the remainder estimate, $R_n \leq \int_n^{\infty} \frac{1}{x^3}\,dx = \frac{1}{2n^2}$

We solve for n so that $\frac{1}{2n^2} < 0.0005$

$$\Rightarrow n^2 > \frac{1}{0.001} = 1000$$

$$\Rightarrow n > \sqrt{1000} \approx 31.6$$

We need $n = 32$ terms to ensure accuracy to within 0.0005

Theorem. If we add s_n to each side of the inequalities for R_n, we obtain:

$$s_n + \int_{n+1}^{\infty} f(x)\,dx \le s \le s_n + \int_{n}^{\infty} f(x)\,dx.$$

These inequalities provide a lower bound and an upper bound for s, giving a more accurate approximation to the sum of the series than the partial sum s_n alone.

Example. Use the above theorem with $n = 10$ to estimate the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$.

The inequalities become

$$s_{10} + \int_{11}^{\infty} \frac{1}{x^3}\,dx \le s \le s_{10} + \int_{10}^{\infty} f(x)\,dx$$

Since $\int_n^{\infty} \frac{1}{x^3}\,dx = \frac{1}{2n^2}$,

$$s_{10} + \frac{1}{2(11)^2} \le s \le s_{10} + \frac{1}{2(10)^2}$$

Using $s_{10} \approx 1.197532$, we get

$$1.201664 \le s \le 1.202532$$

The length of this interval is 0.000868

Note: We can approximate s by the midpoint of this interval. Then the error is at most half of the length of the interval.

Thus: $\sum_{n=1}^{\infty} \frac{1}{n^3} \approx 1.2021$ with error < 0.0005

Remark. If we compare this to the previous example, we see that the improved estimate for s can be much better than the estimate $s \approx s_n$. To make the error smaller than 0.0005, we had to use 32 terms in the previous example, but only 10 terms in the example above.