

§11.2 Practice Problems

True/False. Justify your answer or give a counterexample.

1. If $\sum a_n$ diverges, then a_n must approach infinity.

Solution: False. Counterexample: $a_n = (-1)^n$ does not approach infinity, yet the series diverges.

2. If $\lim_{n \rightarrow \infty} a_n = 0$, then $\sum a_n$ converges.

Solution: False. Counterexample: $a_n = \frac{1}{n}$.

3. If $|r| < 1$, the geometric series $\sum_{n=1}^{\infty} ar^n$ converges to $\frac{a}{1-r}$.

Solution: False. The series converges to $\frac{ar}{1-r}$.

Determine if the series converges or diverges. If it converges, find the limit.

1. $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$

Solution: First, use partial fraction decomposition:

$$\frac{1}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2}.$$

Multiplying both sides by $n(n+2)$, we obtain:

$$1 = A(n+2) + Bn.$$

Setting $n = 0$ gives $1 = 2A$, so $A = \frac{1}{2}$. Setting $n = -2$ gives $1 = -2B$, so $B = -\frac{1}{2}$.

Thus, we rewrite the term:

$$\frac{1}{n(n+2)} = \frac{1/2}{n} - \frac{1/2}{n+2}.$$

Now, consider the partial sum:

$$s_N = \sum_{n=1}^N \left(\frac{1}{2n} - \frac{1}{2(n+2)} \right).$$

Expanding the sum:

$$s_N = \frac{1}{2} \left(\frac{1}{1} - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \cdots + \frac{1}{N} - \frac{1}{N+2} \right).$$

This is a telescoping series, where most terms cancel, leaving:

$$s_N = \frac{1}{2} \left(1 + \frac{1}{2} - \frac{1}{N+1} - \frac{1}{N+2} \right).$$

Taking the limit as $N \rightarrow \infty$,

$$\lim_{N \rightarrow \infty} s_N = \frac{1}{2} \left(1 + \frac{1}{2} - 0 - 0 \right) = \frac{1}{2} \cdot \frac{3}{2} = \frac{3}{4}.$$

Thus, the sum converges to:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \frac{3}{4}.$$

2. $\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n$

Solution: For a geometric series $\sum_{n=0}^{\infty} ar^n$, the sum converges if $|r| < 1$ and equals $\frac{a}{1-r}$. Here, $a = 1$ and $r = -\frac{1}{2}$, so

$$\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n = \frac{1}{1 - \left(-\frac{1}{2}\right)} = \frac{1}{1 + \frac{1}{2}} = \frac{1}{\frac{3}{2}} = \frac{2}{3}.$$

3. $\sum_{n=1}^{\infty} \frac{n^2}{4n^2 + 1}$

Solution: As $n \rightarrow \infty$,

$$\frac{n^2}{4n^2 + 1} \sim \frac{n^2}{4n^2} = \frac{1}{4}.$$

Hence the terms do not go to 0; in fact, they behave like $\frac{1}{4}$ for large n . For a series to converge, its terms must approach 0. Since they do not, the series *diverges* by the Test for Divergence.

4. $\sum_{n=1}^{\infty} \frac{5}{2^n}$

Solution: This is a geometric series with first term $\frac{5}{2}$ (when $n = 1$) and common ratio $\frac{1}{2}$.

$$\sum_{n=1}^{\infty} \left(\frac{5}{2}\right)^n = \frac{\frac{5}{2}}{1 - \frac{1}{2}} = 5.$$

5. $\sum_{n=1}^{\infty} \left(\frac{2}{n(n+1)} + \frac{1}{3^n} \right)$

Solution: Split the series:

$$\sum_{n=1}^{\infty} \frac{2}{n(n+1)} + \sum_{n=1}^{\infty} \frac{1}{3^n}.$$

First part: Use partial fraction decomposition:

$$\frac{2}{n(n+1)} = 2 \left(\frac{1}{n} - \frac{1}{n+1} \right).$$

The partial sum is:

$$s_N = \sum_{n=1}^N 2 \left(\frac{1}{n} - \frac{1}{n+1} \right).$$

Expanding:

$$s_N = 2 \left(\frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{N} - \frac{1}{N+1} \right).$$

This is a telescoping series where most terms cancel, leaving:

$$s_N = 2 \left(1 - \frac{1}{N+1} \right).$$

Taking the limit as $N \rightarrow \infty$,

$$\lim_{N \rightarrow \infty} s_N = 2(1 - 0) = 2.$$

Second part: The second sum is a geometric series:

$$\sum_{n=1}^{\infty} \frac{1}{3^n}.$$

Using the formula for the sum of an infinite geometric series with first term $a = \frac{1}{3}$ and common ratio $r = \frac{1}{3}$:

$$\sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{\frac{1}{3}}{1 - \frac{1}{3}} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}.$$

Adding the results:

$$2 + \frac{1}{2} = \frac{5}{2}.$$

Thus, the sum converges to:

$$\sum_{n=1}^{\infty} \left(\frac{2}{n(n+1)} + \frac{1}{3^n} \right) = \frac{5}{2}.$$

6. $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+1}$

Solution: For a series to converge, its terms must go to 0. Since $\frac{n}{n+1} \rightarrow 1$, the terms do not approach 0. Therefore, the series *diverges* by the Test for Divergence.

7. $\sum_{n=0}^{\infty} 3 \left(\frac{1}{3} \right)^n$

Solution: This is a geometric series with $a = 3$ and ratio $r = \frac{1}{3}$. Thus,

$$\sum_{n=0}^{\infty} \left(\frac{1}{3} \right)^n = \frac{3}{1 - \frac{1}{3}} = \frac{3}{\frac{2}{3}} = \frac{9}{2}.$$

8. $\sum_{n=1}^{\infty} \left(\frac{2^n - 1}{3^n} \right)$

Solution: Split the series:

$$\sum_{n=1}^{\infty} \frac{2^n - 1}{3^n} = \sum_{n=1}^{\infty} \frac{2^n}{3^n} - \sum_{n=1}^{\infty} \frac{1}{3^n} = \sum_{n=1}^{\infty} \left(\frac{2}{3} \right)^n - \sum_{n=1}^{\infty} \left(\frac{1}{3} \right)^n.$$

Each is a geometric series with $|r| < 1$, so both converge. For the first:

$$\sum_{n=1}^{\infty} \left(\frac{2}{3} \right)^n = \frac{\frac{2}{3}}{1 - \frac{2}{3}} = \frac{\frac{2}{3}}{\frac{1}{3}} = 2.$$

For the second:

$$\sum_{n=1}^{\infty} \left(\frac{1}{3} \right)^n = \frac{\frac{1}{3}}{1 - \frac{1}{3}} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}.$$

Hence, the overall sum is

$$2 - \frac{1}{2} = \frac{3}{2}.$$

So it converges and the sum is $\frac{3}{2}$.