11.2 Series

An infinite number of mathematicians walk into a bar, drawn to it because of the bartender's legendary precision. If anyone could handle their unique style of ordering, it was this bartender.

The mathematicians greet the bartender, who immediately pulls out his pen and paper and asks, "What will you all be having today?"

The first mathematician confidently says, "I'll have a beer."

"Sure," says the bartender, jotting down:

$$1$$

The second mathematician chimes in, "I'll have half a beer." The bartender pauses, scratches out the original note, and writes:

$$1+\frac{1}{2}$$

The third mathematician adds, "I'll have a quarter of a beer." The bartender, now suspicious of where this is going, writes:

$$1+\frac{1}{2}+\frac{1}{4}$$

This continues, with each mathematician ordering half of what the previous one did. The bartender's paper quickly fills up with calculations:

n	Order	Total Beer
1	1	1
2	$1+\frac{1}{2}$	1.5
3	$1+\frac{1}{2}+\frac{1}{4}$	1.75
4	$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}$	1.875
5	$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}$	1.9375
6	$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}$	1.96875
7	$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}$	1.984375
8	$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\frac{1}{128}$	1.9921875
$\vdots$	$\vdots$	$\vdots$
50	$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^{49}}$	1.9999999999999991
$\vdots$	$\vdots$	$\vdots$
99	$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^{98}}$	1.99999999999999999999999999999

Before the 100th mathematician can speak, the bartender slams two beers on the counter and yells, "Enough! You all need to learn your limits."

Key Idea: The partial sums form a sequence: $S_1, S_2, S_3, S_4, \ldots$

$a_1 \quad a_1+a_2 \quad a_1+a_2+a_3 \quad a_1+a_2+a_3+a_4$

and we can decide whether a sequence converges or diverges.

Definition. A **series** is the sum of the terms of a sequence. Suppose we have a sequence of numbers $a_1, a_2, a_3, \ldots$. The corresponding series is written as:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots .$$

To make sense of this infinite sum, we do not add all the terms at once. Instead, we define the sum of the series as the **limit of the partial sums**. The nth partial sum S_n is given by:

$$S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n.$$

If the sequence of partial sums $\{S_n\}$ approaches a finite limit L as $n \to \infty$, we say that the series **converges**, and we write:

$$\sum_{n=1}^{\infty} a_n = L.$$

If the sequence of partial sums does not approach a finite limit, we say the series **diverges**.

Example. Suppose we know that the sum of the first n terms of the series $\sum_{n=1}^{\infty} a_n$ is

$$s_n = a_1 + a_2 + \cdots + a_n = \frac{2n}{3n+5}.$$

What is the sum of the series?

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{2n}{3n+5} \cdot \frac{1/n}{1/n} = \lim_{n\to\infty} \frac{2}{3+\frac{5}{n}} = \frac{2}{3}$$

Example. Consider the series:

$$1 + 2 + 3 + 4 + \cdots$$

Intuitively, this sum diverges. But why?

$$s_n = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

$\lim_{n\to\infty} s_n = \infty$, So the sequence of partial sums s_n grows without bound and the sum diverges.

Example. Show that the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent, and find its sum.

$$s_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)}$$

Key observation: $\frac{1}{i(i+1)} = \frac{1}{i} - \frac{1}{i+1}$

$$s_n = \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1}$$

Now, $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1 - 0 = 1$

Hence $\sum_{i=1}^{\infty} \frac{1}{n(n+1)} = 1$

Sum of a Geometric Series

Definition. A **geometric series** is a series in which each term is obtained by multiplying the previous term by a constant ratio r:

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots$$

If $|r| < 1$, the geometric series is **convergent**, and its sum is $\frac{a}{1-r}$. $\frac{\text{first term}}{1-\text{common ratio}}$

If $|r| \geq 1$, the geometric series is **divergent**.

Proof.

- If $r = 1$, what is the nth partial sum s_n?

  $s_n = a + a + \cdots + a = na$. As $n \to \infty$, $s_n \to \infty$, so the series diverges.

- If $r = -1$, what is the nth partial sum s_n?

  $s_n = a - a + a - a + \cdots \pm a$. If n is even, $s_n = 0$. If n is odd, $s_n = a$.
  As $n \to \infty$, s_n oscillates between 0 and a, so the series diverges.

- If $r \neq 1$, what is the nth partial sum s_n?

  $$s_n = a + ar + ar^2 + \cdots + ar^{n-1}$$
  $$rs_n = ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n$$
  $$s_n - rs_n = a - ar^n$$
  $$s_n(1-r) = a(1-r^n) \Rightarrow s_n = \frac{a(1-r^n)}{1-r}$$

  We can divide by $1-r$ since $r \neq 1$

- Take the limit of s_n as $n \to \infty$ if $-1 < r < 1$.

  Since $r^n \to 0$ as $n \to \infty$, $\lim_{n\to\infty} s_n = \frac{a(1-0)}{1-r} = \frac{a}{1-r}$

- Take the limit of s_n as $n \to \infty$ if $|r| > 1$.

  As $n \to \infty$, r^n diverges. Hence s_n diverges as $n \to \infty$.

□

Example. Find the sum of the geometric series

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots.$$

The first term is $a = 5$

The common ratio is $r = -\frac{2}{3}$

Since $|r| = \frac{2}{3} < 1$, the series converges

The sum is $\frac{a}{1-r} = \frac{5}{1-(-\frac{2}{3})} = \frac{5}{1+\frac{2}{3}} = \frac{5}{\frac{5}{3}} = 3$

Example. Is the series $\sum_{n=1}^{\infty} 2^{2n} 3^{1-n}$ convergent or divergent?

Goal: rewrite as $\sum_{n=1}^{\infty} a r^{n-1}$

$$\sum_{n=1}^{\infty} 2^{2n} \cdot 3^{1-n} = \sum_{n=1}^{\infty} (2^2)^n \cdot 3^{-(n-1)} = \sum_{n=1}^{\infty} \frac{4^n}{3^{n-1}} = \sum_{n=1}^{\infty} 4 \cdot \frac{4^{n-1}}{3^{n-1}} = \sum_{n=1}^{\infty} 4 \cdot \left(\frac{4}{3}\right)^{n-1}$$

This is a geometric series with $a = 4$ and $r = \frac{4}{3}$

Since $r > 1$, the series diverges

Example. Find the sum of the series $\sum_{n=0}^{\infty} x^n$, where $|x| < 1$.

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \ldots$$

The first term is 1

Since we adopt the convention that $x^0 = 1$ even when $x = 0$

This is a geometric series with $a = 1$ and $r = x$

Since $|r| = |x| < 1$, it converges, and the sum is given by

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

Preview of Taylor Series

Test for Divergence

Example. Show that the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

is divergent.

Consider the partial sums $s_2, s_4, s_8, s_{16}, s_{32}, \ldots$ and show that they become large.

$$s_2 = 1 + \frac{1}{2}$$

$$\begin{aligned} s_4 &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) \\ &> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) \\ &= 1 + \frac{2}{2} \end{aligned}$$

$$\begin{aligned} s_8 &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) \\ &> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) \\ &= 1 + \frac{3}{2} \end{aligned}$$

Similarly, $s_{16} > 1 + \frac{4}{2}$ and $s_{32} > 1 + \frac{5}{2}$. In general, $s_{2^n} > 1 + \frac{n}{2}$. This shows that $s_{2^n} \to \infty$, so $\{s_{2^n}\}$ is divergent. Hence $\{s_n\}$ is divergent. In particular, the harmonic series divergence.

Theorem. If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

Let $s_n = a_1 + a_2 + a_3 + \dots + a_n$

Since $\sum_{n=1}^{\infty} a_n$ converges, the sequence $\{s_n\}$ is convergent

Let $\lim_{n\to\infty} s_n = s$. Note that $\lim_{n\to\infty} s_{n-1} = s$.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = s - s = 0$$

Test for Divergence

Corollary. If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Example. True or False: If $\lim_{n\to\infty} a_n = 0$, then $\sum a_n$ is convergent.

False. The harmonic series diverges, but $\lim_{n\to\infty} \frac{1}{n} = 0$.

Example. Show that the series $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ diverges.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{5n^2+4} \cdot \frac{1/n^2}{1/n^2} = \lim_{n\to\infty} \frac{1}{5 + 4/n^2} = \frac{1}{5} \neq 0$$

Since $\lim_{n\to\infty} a_n \neq 0$, the series diverges by the Test for Divergence.

Properties of Convergent Series

Theorem. If $\sum a_n$ and $\sum b_n$ are convergent series, then so are the series $\sum ca_n$ (where c is a constant), $\sum(a_n + b_n)$, and $\sum(a_n - b_n)$, and:

(i) $\displaystyle\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n,$

(ii) $\displaystyle\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n,$

(iii) $\displaystyle\sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n.$

Example. Find the sum of the series

$$\sum_{n=1}^{\infty}\left(\frac{3}{n(n+1)} + \frac{1}{2^n}\right).$$

The series $\sum \frac{1}{2^n}$ is a geometric series with $a = 1/2$ and $r = 1/2$

So $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{a}{1-r} = \frac{1/2}{1-1/2} = 1$

Also, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$ (see example we did).

So, $\displaystyle\sum_{n=1}^{\infty}\left(\frac{3}{n(n+1)} + \frac{1}{2^n}\right) = 3 \cdot \sum_{n=1}^{\infty} \frac{1}{n(n+1)} + \sum_{n=1}^{\infty} \frac{1}{2^n}$

$= 3 \cdot 1 + 1$

$= 4$