11.11 Taylor Series Remainder Estimate

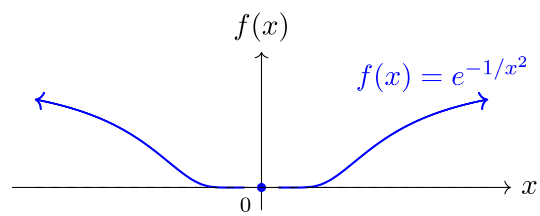
When we approximate a function using a Taylor polynomial, we leave out the rest of the infinite series. The difference between the function and the polynomial is called the **remainder**, and understanding this error is crucial for determining the accuracy of our approximation. The **Taylor remainder estimate** gives a bound on this error and allows us to guarantee how close the Taylor polynomial is to the true function value.

This estimate is especially important when we use Taylor polynomials for numerical approximation, such as estimating values like e^x, $\sin x$, or $\ln(1+x)$, where knowing how accurate the approximation is can guide how many terms we need.

Example. Let

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Show that the Maclaurin series for $f(x)$ does not represent the function on any interval around 0. The series and the function only agree at the point $x = 0$.



Question. The function $f(x)$ in the previous example has all derivatives equal to 0 at $x = 0$, so its Maclaurin series is the zero series. But for $x \neq 0$, $f(x)$ is positive, so the Taylor series does *not* equal the function. This leads to an important question: *When does a Taylor series actually equal the function?* The following theorem answers this question.

Theorem (Taylor Remainder)**.** Let $f(x)$ be a function with an nth-degree Taylor polynomial $T_n(x)$ centered at a. Define the **remainder** as:

$$R_n(x) = f(x) - T_n(x).$$

If $\lim\limits_{n\to\infty} R_n(x) = 0$ for $|x-a| < R$, then f is equal to the sum of its Taylor series on the interval $|x-a| < R$.

In trying to show that $\lim_{n\to\infty} R_n = 0$ for a specific function $f(x)$, we usually use the following theorem.

Theorem (Taylor's Inequality)**.** If $\left|f^{(n+1)}(x)\right| \leq M$ for $|x-a| \leq d$, then the remainder $R_n(x)$ of the Taylor series satisfies

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \quad \text{for } |x-a| \leq d.$$

Example. Prove that e^x is equal to the sum of its Maclaurin series.

Example. Recall the Maclaurin series for $\sin x$:

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Prove that this represents $\sin x$ for all x.

Example. Find the Taylor series for $f(x) = \sin x$ centered at $a = \frac{\pi}{3}$ and prove that this represents $f(x) = \sin x$.

When using a Taylor polynomial T_n to approximate a function f, we have to ask the questions:

- How good an approximation is it?
- How large should we take n to be in order to achieve a desired accuracy?

To answer these questions, we look at the absolute value of the remainder:

$$|R_n(x)| = |f(x) - T_n(x)|.$$

There are three possible methods for estimating the size of the error:

1. Graph $|R_n(x)| = |f(x) - T_n(x)|$ using a calculator or computer.
2. If the series is alternating, use the Alternating Series Estimation Theorem.
3. Use Taylor's Inequality: If $|f^{(n+1)}(x)| \leq M$, then $|R_n(x)| \leq \dfrac{M}{(n+1)!}|x-a|^{n+1}$.

Example.

(a) Approximate $f(x) = \sqrt[3]{x}$ by a Taylor polynomial of degree 2 at $a = 8$.

(b) How accurate is this approximation when $7 < x < 9$?

Example.

(a) What is the maximum error possible in using the approximation

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

when $-0.3 < x < 0.3$? Use this approximation to find $\sin(12^\circ)$ correct to six decimal places.

(b) For what values of x is this approximation accurate to within 0.00005?

Example.

(a) Approximate $f(x) = x^{1/4}$ by a degree 3 Taylor polynomial at $a = 1$.

(b) Use Taylor's Inequality to estimate the error when $x \in [0.95, 1.05]$.

Example.

(a) Approximate $f(x) = \ln(1 + 3x)$ by a degree 3 Taylor polynomial centered at $a = 2$.

(b) Use Taylor's Inequality to estimate the error $|R_3(x)|$ on the interval $[1.8, 2.2]$. Round your answer to six decimal places.

Example. Use Taylor's Inequality to determine the minimum number of terms of the Maclaurin series for e^x needed to estimate $e^{0.2}$ to within 0.000001.