11.11 Taylor Series Remainder Estimate

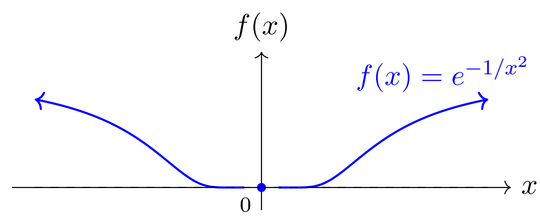
When we approximate a function using a Taylor polynomial, we leave out the rest of the infinite series. The difference between the function and the polynomial is called the **remainder**, and understanding this error is crucial for determining the accuracy of our approximation. The **Taylor remainder estimate** gives a bound on this error and allows us to guarantee how close the Taylor polynomial is to the true function value.

This estimate is especially important when we use Taylor polynomials for numerical approximation, such as estimating values like e^x, $\sin x$, or $\ln(1+x)$, where knowing how accurate the approximation is can guide how many terms we need.

Example. Let

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Show that the Maclaurin series for $f(x)$ does not represent the function on any interval around 0. The series and the function only agree at the point $x = 0$.



- You can show using the limit definition of the derivative that $f^{(n)}(0) = 0$ for all n (the function is extremely flat near 0)
- Hence the Maclaurin series is $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = 0 + 0x + 0x^2 + \dots$
- But $f(x) \neq 0$, except at $x=0$. So the Taylor Series only matches the function at $x=0$.
- Conclusion: just because a Taylor Series converges to something, doesn't mean it converges to the function.
- Therefore, we need something like the Taylor remainder estimate to help check whether the polynomial approximates the function well.

Question. The function $f(x)$ in the previous example has all derivatives equal to 0 at $x = 0$, so its Maclaurin series is the zero series. But for $x \neq 0$, $f(x)$ is positive, so the Taylor series does *not* equal the function. This leads to an important question: *When does a Taylor series actually equal the function?* The following theorem answers this question.

Theorem (Taylor Remainder)**.** Let $f(x)$ be a function with an nth-degree Taylor polynomial $T_n(x)$ centered at a. Define the **remainder** as:

$$R_n(x) = f(x) - T_n(x).$$

If $\lim_{n\to\infty} R_n(x) = 0$ for $|x-a| < R$, then f is equal to the sum of its Taylor series on the interval $|x-a| < R$.

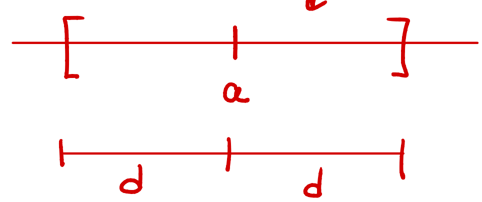
In trying to show that $\lim_{n\to\infty} R_n = 0$ for a specific function $f(x)$, we usually use the following theorem.

We want to bound $|f^{(n+1)}(x)|$ by M on this interval

Theorem (Taylor's Inequality)**.** If $\left|f^{(n+1)}(x)\right| \leq M$ for $|x-a| \leq d$, then the remainder $R_n(x)$ of the Taylor series satisfies

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \quad \text{for } |x-a| \leq d.$$

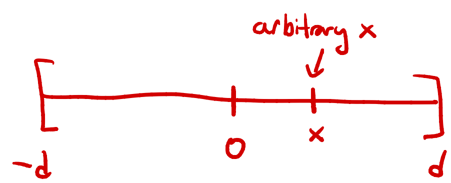
Example. Prove that e^x is equal to the sum of its Maclaurin series.



a real number x

- Fix $x \in \mathbb{R}$. We want to show $R_n(x) \to 0$
- Let d be any positive number with $|x| \leq d$, then $|f^{(n+1)}(x)| = e^x \leq e^d$
- Apply Taylor's Inequality with $a = 0$ and $M = e^d$:

$$|R_n(x)| \leq \frac{e^d}{(n+1)!}|x|^{n+1}$$



- Hence

$$0 \leq \lim_{n\to\infty} |R_n(x)| \leq \lim_{n\to\infty} \frac{e^d}{(n+1)!}|x|^{n+1} = e^d \cdot \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$$

- Conclude: since x was arbitrary, e^x equals the sum of its Maclaurin series for all x.

Example. Recall the Maclaurin series for $\sin x$:

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Prove that this represents $\sin x$ for all x.

- Fix any $x \in \mathbb{R}$. We want to use Taylor's Inequality to show $R_n(x) \to 0$
- The derivatives of $\sin x$ are either $\pm \sin x$ or $\pm \cos x$, all of which are bounded by 1 in absolute value.
- Take $M = 1$ and apply Taylor's inequality with $a = 0$:

$$|R_n(x)| \leq \frac{M}{(n+1)!} |x|^{n+1} = \frac{|x|^{n+1}}{(n+1)!}$$

- As $n \to \infty$, $\quad 0 \leq \lim_{n\to\infty} |R_n(x)| \leq \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$
- By the Squeeze theorem, $R_n(x) \to 0$ (since $-|R(x)| \leq R_n(x) \leq |R_n(x)|$)
- By the Taylor Remainder Theorem, $\sin x$ equals the sum of its Maclaurin series for all x.

HW Question

Example. Find the Taylor series for $f(x) = \sin x$ centered at $a = \frac{\pi}{3}$ and prove that this represents $f(x) = \sin x$.

We compute derivatives and evaluate them at $x = \frac{\pi}{3}$

$f(x) = \sin x$ $\qquad$ $f(\pi/3) = \sqrt{3}/2$

$f'(x) = \cos x$ $\qquad$ $f'(\pi/3) = 1/2$

$f''(x) = -\sin x$ $\qquad$ $f''(\pi/3) = -\sqrt{3}/2$

$f'''(x) = -\cos x$ $\qquad$ $f'''(\pi/3) = -1/2$

The pattern repeats every four terms: $\sqrt{3}/2, 1/2, -\sqrt{3}/2, -1/2$

The Taylor Series is:

$$f(x) = f(\pi/3) + f'(\pi/3)(x-\pi/3) + \frac{f''(\pi/3)}{2!}(x-\pi/3)^2 + \frac{f'''(\pi/3)}{3!}(x-\pi/3)^3 + \cdots$$

$$= \sqrt{3}/2 + \frac{1}{2}(x-\pi/3) - \frac{\sqrt{3}}{2}\cdot\frac{1}{2!}(x-\pi/3)^2 - \frac{1}{2}\cdot\frac{1}{3!}(x-\pi/3)^3 + \cdots$$

Even Terms + odd Terms

$$= \sum_{n=0}^{\infty} \frac{(-1)^n\sqrt{3}}{2(2n)!}\left(x-\frac{\pi}{3}\right)^{2n} + \sum_{n=0}^{\infty} \frac{(-1)^n}{2(2n+1)!}\left(x-\frac{\pi}{3}\right)^{2n+1}$$

(2) Fix $x \in \mathbb{R}$. We can use Taylor's Inequality to show $R_n(x) \to 0$

Since the derivatives of $\sin x$ are $\pm\sin x$ or $\pm\cos x$, take $M = 1$ and $a = \pi/3$:

$$|R_n(x)| \leq \frac{M}{(n+1)!}\left|x-\frac{\pi}{3}\right|^{n+1} = \frac{\left|x-\frac{\pi}{3}\right|^{n+1}}{(n+1)!}$$

As $n\to\infty$, $$0 \leq \lim_{n\to\infty}|R_n(x)| \leq \lim_{n\to\infty}\frac{|x-\pi/3|^{n+1}}{(n+1)!} = 0$$

We conclude $R_n(x) \to 0$ and so this Taylor Series represents $\sin x$ for all x.

When using a Taylor polynomial T_n to approximate a function f, we have to ask the questions:

- How good an approximation is it?
- How large should we take n to be in order to achieve a desired accuracy?

To answer these questions, we look at the absolute value of the remainder:

$$|R_n(x)| = |f(x) - T_n(x)|.$$

There are three possible methods for estimating the size of the error:

1. Graph $|R_n(x)| = |f(x) - T_n(x)|$ using a calculator or computer.
2. If the series is alternating, use the Alternating Series Estimation Theorem.
3. Use Taylor's Inequality: If $|f^{(n+1)}(x)| \le M$, then $|R_n(x)| \le \dfrac{M}{(n+1)!}|x-a|^{n+1}$.

Example.

(a) Approximate $f(x) = \sqrt[3]{x}$ by a Taylor polynomial of degree 2 at $a = 8$.

(b) How accurate is this approximation when $7 < x < 9$?

(a) $f(x) = x^{1/3}$ $\qquad f(8) = 2$

$f'(x) = \frac{1}{3}x^{-2/3}$ $\qquad f'(8) = \frac{1}{12}$

$f''(x) = \frac{-2}{9}x^{-5/3}$ $\qquad f''(8) = -\frac{1}{144}$

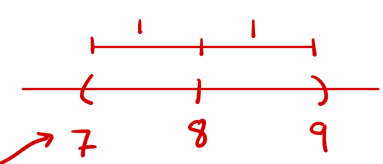
$\Rightarrow$ $T_2(x) = f(8) + f'(8)(x-8) + \frac{f''(8)}{2!}(x-8)^2$

$T_2(x) = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2$

(b) Use Taylor's Inequality with $n=2$ and $a=8$

This function is decreasing

We need to bound $|f^{(3)}(x)| = \left|\frac{10}{27}x^{-8/3}\right|$



For $x > 7$, $|f^{(3)}(x)| < \frac{10}{27 \cdot 7^{8/3}} \approx 0.0021$

Take $M = 0.0021$. Then for $7 < x < 9$

$$|R_2(x)| < \frac{M}{(n+1)!}|x-a|^{n+1} = \frac{0.0021}{3!}|x-8|^3 < \frac{0.0021}{3!}(1)^3 \approx 0.0004$$

Conclude: the approximation is accurate to within 0.0004 on $7 < x < 9$

Example.

(a) What is the maximum error possible in using the approximation

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

when $-0.3 < x < 0.3$? Use this approximation to find $\sin(12^\circ)$ correct to six decimal places.

(b) For what values of x is this approximation accurate to within 0.00005?

(a) • The Maclaurin series $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ is alternating for all nonzero values of x, and the terms both decrease in size and go to 0 on $-0.3 < x < 0.3$ since $|x| < 1$. The Alternating Series Estimate applies.

• The error is at most $\left| \frac{x^7}{7!} \right| = \frac{|x|^7}{5040}$

• If $-0.3 < x < 0.3$, then $|x| \leq 0.3$ so the error is at most $\frac{(0.3)^7}{5040} \approx 4.3 \times 10^{-8}$

• $\sin(12^\circ) = \sin\left(\frac{\pi}{15}\right) \approx \frac{\pi}{15} - \left(\frac{\pi}{15}\right)^3 \cdot \frac{1}{3!} + \left(\frac{\pi}{15}\right)^5 \cdot \frac{1}{5!} \approx 0.207912$

This is correct to at least 6 decimal places by our error bound.

(b) The error is smaller than 0.00005 if

$$\frac{|x|^7}{5040} < 0.00005 \Rightarrow |x|^7 < 0.252 \Rightarrow |x| < (0.252)^{1/7} \approx 0.82$$

Conclude: if $-0.82 < x < 0.82$, the error is less than 0.00005

Example.

(a) Approximate $f(x) = x^{1/4}$ by a degree 3 Taylor polynomial at $a = 1$.

(b) Use Taylor's Inequality to estimate the error when $x \in [0.95, 1.05]$.

Example.

(a) Approximate $f(x) = \ln(1 + 3x)$ by a degree 3 Taylor polynomial centered at $a = 2$.

(b) Use Taylor's Inequality to estimate the error $|R_3(x)|$ on the interval $[1.8, 2.2]$. Round your answer to six decimal places.

Example. Use Taylor's Inequality to determine the minimum number of terms of the Maclaurin series for e^x needed to estimate $e^{0.2}$ to within 0.000001.

- The Maclaurin series for $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

- Taylor's Inequality says $|R_n(x)| \leq \frac{M}{(n+1)!}|x|^{n+1}$

  where $M \geq |f^{(n+1)}(x)|$ on the interval $[-0.2, 0.2]$

- Since $f^{(n+1)}(x) = e^x$ and e^x is increasing, so take $M = e^{0.2} \approx 1.221$

- We want to find n so that

$$|R_n(x)| < \frac{M}{(n+1)!}|x|^{n+1} < \frac{1.221}{(n+1)!}(0.2)^{n+1} < 0.000001$$

- Try successive values of n:

$n=3$: $\frac{1.221}{4!}(0.2)^4 \approx 0.0000815$ ✗

$n=4$: $\frac{1.221}{5!}(0.2)^5 \approx 0.00000326$ ✗

$n=5$: $\frac{1.221}{6!}(0.2)^6 \approx 0.000000109$ ✓

So, the error is less than 10^{-6} when $n=5$. This means we need the first 6 terms of the Maclaurin series (since the series starts at $n=0$).