11.10 Taylor and Maclaurin Series

Taylor polynomials approximate functions locally by matching the value and derivatives of a function at a fixed point. While these polynomials are useful for short-range approximation, they do not always capture the full behavior of a function. In this section, we explore Taylor series, which extend Taylor polynomials to infinite degree. These series allow us to represent functions globally (within a radius of convergence) using an infinite sum of derivatives.

Definition. If a function f has derivatives of all orders at a point a, its **Taylor series centered at** a is:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

The Taylor series centered at $a = 0$ is called the **Maclaurin series**:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

Common Maclaurin Series

The following are standard Maclaurin series for important functions. You are expected to memorize these and be able to use them to generate new series by substitution, differentiation, or integration.

Function	Maclaurin Series	Radius of Convergence
e^x	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	∞
$\sin x$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$	∞
$\cos x$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$	∞
$\frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n$	1
$\ln(1+x)$	$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$	1
$\arctan x$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$	1

Theorem. Suppose a function f can be written as a power series centered at a:

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

for all x in some interval about a. Then this power series is the Taylor series for f centered at a. In particular,

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

Example. Find the Taylor series for $f(x) = \dfrac{1}{x}$ centered at $a = 1$.

Example. Find the Maclaurin series for $f(x) = \arctan(x^2)$.

Example. Find the sum of the series:

$$\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{4^{2n}(2n)!}$$