Taylor Polynomials (Solutions)

For each function,

$$T_4(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4.$$

1. $f(x) = x^2 e^x, \quad a = 0$

$$\begin{aligned} f(x) &= x^2 e^x & f(0) &= 0 \\ f'(x) &= (2x + x^2)e^x & f'(0) &= 0 \\ f''(x) &= (2 + 4x + x^2)e^x & f''(0) &= 2 \\ f'''(x) &= (6 + 6x + x^2)e^x & f'''(0) &= 6 \\ f^{(4)}(x) &= (12 + 8x + x^2)e^x & f^{(4)}(0) &= 12 \end{aligned}$$

$$T_4(x) = 0 + 0x + \frac{2}{2!}x^2 + \frac{6}{3!}x^3 + \frac{12}{4!}x^4 = x^2 + x^3 + \frac{x^4}{2}.$$

2. $f(x) = \dfrac{1}{1+x^2}, \quad a = 1$

$$\begin{aligned} f(x) &= (1+x^2)^{-1} & & \\ f'(x) &= -\frac{2x}{(1+x^2)^2} & f(1) &= \frac{1}{2} \\ f''(x) &= \frac{6x^2 - 2}{(1+x^2)^3} & f'(1) &= -\frac{1}{2} \\ f'''(x) &= \frac{24x(1-x^2)}{(1+x^2)^4} & f''(1) &= \frac{1}{2} \\ & & f'''(1) &= 0 \\ f^{(4)}(x) &= \frac{24(1 - 10x^2 + 5x^4)}{(1+x^2)^5} & f^{(4)}(1) &= -3 \end{aligned}$$

$$T_4(x) = \frac{1}{2} - \frac{1}{2}(x-1) + \frac{1/2}{2!}(x-1)^2 + \frac{0}{3!}(x-1)^3 + \frac{-3}{4!}(x-1)^4.$$

$$T_4(x) = \frac{1}{2} - \frac{1}{2}(x-1) + \frac{1}{4}(x-1)^2 - \frac{1}{8}(x-1)^4.$$

3. $f(x) = \cos(x^2), \quad a = 0$

$$\begin{aligned} f(x) &= \cos(x^2) & f(0) &= 1 \\ f'(x) &= -2x\sin(x^2) & f'(0) &= 0 \\ f''(x) &= -2\sin(x^2) - 4x^2\cos(x^2) & f''(0) &= 0 \\ f'''(x) &= -12x\cos(x^2) + 8x^3\sin(x^2) & f'''(0) &= 0 \\ f^{(4)}(x) &= -12\cos(x^2) + 48x^2\sin(x^2) + 16x^4\cos(x^2) & f^{(4)}(0) &= -12 \end{aligned}$$

$$T_4(x) = 1 + 0x + \frac{0}{2!}x^2 + \frac{0}{3!}x^3 + \frac{-12}{4!}x^4 = 1 - \frac{x^4}{2}.$$

4. $f(x) = \sin(2x), \quad a = \frac{\pi}{4}$

$$\begin{aligned} f(x) &= \sin(2x) & f\left(\frac{\pi}{4}\right) &= 1 \\ f'(x) &= 2\cos(2x) & f'\left(\frac{\pi}{4}\right) &= 0 \\ f''(x) &= -4\sin(2x) & f''\left(\frac{\pi}{4}\right) &= -4 \\ f'''(x) &= -8\cos(2x) & f'''\left(\frac{\pi}{4}\right) &= 0 \\ f^{(4)}(x) &= 16\sin(2x) & f^{(4)}\left(\frac{\pi}{4}\right) &= 16 \end{aligned}$$

$$T_4(x) = 1 + 0\left(x - \frac{\pi}{4}\right) + \frac{-4}{2!}\left(x - \frac{\pi}{4}\right)^2 + \frac{0}{3!}\left(x - \frac{\pi}{4}\right)^3 + \frac{16}{4!}\left(x - \frac{\pi}{4}\right)^4.$$

$$T_4(x) = 1 - 2\left(x - \frac{\pi}{4}\right)^2 + \frac{2}{3}\left(x - \frac{\pi}{4}\right)^4.$$

5. $f(x) = \ln(1 + 2x), \quad a = 0$

$$\begin{aligned} f(x) &= \ln(1+2x) & & \\ f'(x) &= \frac{2}{1+2x} & f(0) &= 0 \\ f''(x) &= -\frac{4}{(1+2x)^2} & f'(0) &= 2 \\ f'''(x) &= \frac{16}{(1+2x)^3} & f''(0) &= -4 \\ f^{(4)}(x) &= -\frac{96}{(1+2x)^4} & f'''(0) &= 16 \\ & & f^{(4)}(0) &= -96 \end{aligned}$$

$$T_4(x) = 0 + 2x + \frac{-4}{2!}x^2 + \frac{16}{3!}x^3 + \frac{-96}{4!}x^4 = 2x - 2x^2 + \frac{8}{3}x^3 - 4x^4.$$

6. $f(x) = \tan^{-1}(x), \quad a = 0$

$$\begin{aligned} f(x) &= \tan^{-1}(x) & & \\ f'(x) &= \frac{1}{1+x^2} & f(0) &= 0 \\ f''(x) &= -\frac{2x}{(1+x^2)^2} & f'(0) &= 1 \\ f'''(x) &= \frac{6x^2-2}{(1+x^2)^3} & f''(0) &= 0 \\ f^{(4)}(x) &= \frac{24x(1-x^2)}{(1+x^2)^4} & f'''(0) &= -2 \\ & & f^{(4)}(0) &= 0 \end{aligned}$$

$$T_4(x) = 0 + x + \frac{0}{2!}x^2 + \frac{-2}{3!}x^3 + \frac{0}{4!}x^4 = x - \frac{x^3}{3}.$$

7. $f(x) = \dfrac{x}{1-x}, \quad a = 0$

$$\begin{aligned} f(x) &= \frac{x}{1-x} \\ f'(x) &= \frac{1}{(1-x)^2} \\ f''(x) &= \frac{2}{(1-x)^3} \\ f'''(x) &= \frac{6}{(1-x)^4} \\ f^{(4)}(x) &= \frac{24}{(1-x)^5} \end{aligned} \qquad \begin{aligned} f(0) &= 0 \\ f'(0) &= 1 \\ f''(0) &= 2 \\ f'''(0) &= 6 \\ f^{(4)}(0) &= 24 \end{aligned}$$

$$T_4(x) = 0 + x + \frac{2}{2!}x^2 + \frac{6}{3!}x^3 + \frac{24}{4!}x^4 = x + x^2 + x^3 + x^4.$$

8. $f(x) = e^{x^2}, \quad a = 0$

$$\begin{aligned} f(x) &= e^{x^2} \\ f'(x) &= 2xe^{x^2} \\ f''(x) &= (2+4x^2)e^{x^2} \\ f'''(x) &= (12x+8x^3)e^{x^2} \\ f^{(4)}(x) &= (12+48x^2+16x^4)e^{x^2} \end{aligned} \qquad \begin{aligned} f(0) &= 1 \\ f'(0) &= 0 \\ f''(0) &= 2 \\ f'''(0) &= 0 \\ f^{(4)}(0) &= 12 \end{aligned}$$

$$T_4(x) = 1 + 0x + \frac{2}{2!}x^2 + \frac{0}{3!}x^3 + \frac{12}{4!}x^4 = 1 + x^2 + \frac{x^4}{2}.$$

9. $f(x) = \dfrac{1}{\sqrt{1-x}}, \quad a = 0$

$$\begin{aligned} f(x) &= (1-x)^{-1/2} \\ f'(x) &= \frac{1}{2}(1-x)^{-3/2} \\ f''(x) &= \frac{3}{4}(1-x)^{-5/2} \\ f'''(x) &= \frac{15}{8}(1-x)^{-7/2} \\ f^{(4)}(x) &= \frac{105}{16}(1-x)^{-9/2} \end{aligned} \qquad \begin{aligned} f(0) &= 1 \\ f'(0) &= \frac{1}{2} \\ f''(0) &= \frac{3}{4} \\ f'''(0) &= \frac{15}{8} \\ f^{(4)}(0) &= \frac{105}{16} \end{aligned}$$

$$T_4(x) = 1 + \frac{1}{2}x + \frac{3/4}{2!}x^2 + \frac{15/8}{3!}x^3 + \frac{105/16}{4!}x^4.$$

$$T_4(x) = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \frac{35}{128}x^4.$$