11.10 Taylor and Maclaurin Polynomials

Taylor polynomials approximate functions locally using finite-degree polynomials. They simplify functions and approximate their behavior near a point. They match a function's value and derivatives at a given point, making them invaluable in physics, engineering, and numerical computation.

Definition. The **Taylor polynomial** of degree n for a function $f(x)$ centered at a is given by:

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

If the Taylor polynomial is centered at $a = 0$, it is called a **Maclaurin polynomial**:

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n.$$

Example. Prove that the nth degree Taylor polynomial $P_n(x)$ of a function $f(x)$, centered at a, has the same value and the same derivatives, up to order n, as $f(x)$ at $x = a$.

key idea

(also, why we need factorials)

- Step 1: Let $k \leq n$. Compute the k-th derivative of the Taylor polynomial.

Terms with degree less than k will vanish. What remains is:

$$P_n^{(k)}(x) = \frac{f^{(k)}(a)}{k!} \cdot k! + \text{"terms with } (x-a) \text{ in them"}$$

- Step 2: Evaluate the k-th derivative at $x = a$.

At $x=a$, all terms involving $(x-a)$ vanish. Thus

$$P_n^{(k)}(x) = f^{(k)}(a) \checkmark$$

✱ You did this yesterday on the activity.

Example. Compute the 4th-degree Taylor polynomial for $f(x) = e^x$ centered at $a = 0$. Graph $f(x) = e^x$ and its 1st-, 2nd- and 3rd-degree Taylor polynomials to observe local accuracy near $a = 0$.

① $P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$

②
$f(x) = e^x \qquad f(0) = 1$

$f'(x) = e^x \qquad f'(0) = 1$

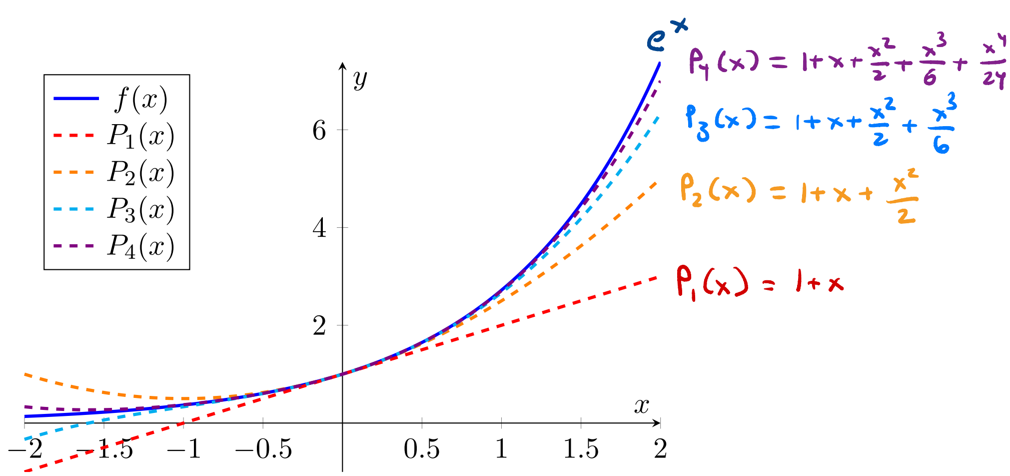
$f''(x) = e^x \quad \Rightarrow \quad f''(0) = 1$

$f'''(x) = e^x \qquad f'''(0) = 1$

$f^{(4)}(x) = e^x \qquad f^{(4)}(0) = 1$

③ $P_4(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$

$$\boxed{P_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}}$$



Example. Find the 3rd-degree Taylor polynomial for $f(x) = \sin(x)$, centered at $a = 0$.

① $P_3(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3$

② $f(x) = \sin(x)$ $f(0) = \sin(0) = 0$

$f'(x) = \cos(x)$ $\Rightarrow$ $f'(0) = \cos(0) = 1$

$f''(x) = -\sin(x)$ $f''(0) = -\sin(0) = 0$

$f'''(x) = -\cos(x)$ $f'''(0) = -\cos(0) = -1$

③ $P_3(x) = 0 + x + \frac{0}{2!} x^2 - \frac{1}{3!} x^3$

$$\boxed{P_3(x) = x - \frac{x^3}{6}}$$

Example. Find the 3rd-degree Taylor polynomial for $f(x) = \frac{1}{x}$, centered at $a = 2$.

① $P_3(x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3$

② $f(x) = \frac{1}{x} = x^{-1}$ $\qquad f(2) = \frac{1}{2}$

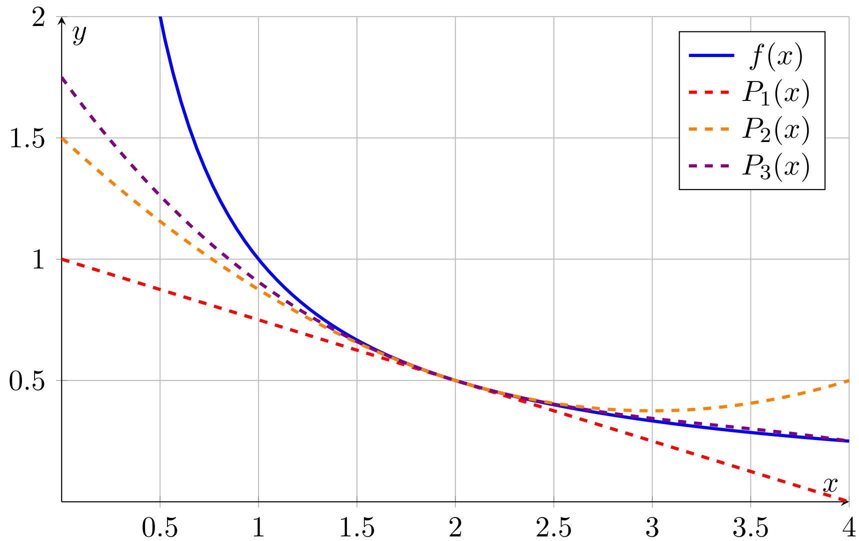
$f'(x) = -x^{-2} = \frac{-1}{x^2}$ $\quad \Rightarrow \quad f'(2) = -\frac{1}{4}$

$f''(x) = 2x^{-3} = \frac{2}{x^3}$ $\qquad f''(2) = \frac{2}{8} = \frac{1}{4}$

$f'''(x) = -6x^{-4} = \frac{-6}{x^4}$ $\qquad f'''(2) = \frac{-6}{16} = \frac{-3}{8}$

③ $P_3(x) = \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1/4}{2!}(x-2)^2 - \frac{3/8}{3!}(x-2)^3$

$$P_3(x) = \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{8}(x-2)^2 - \frac{1}{16}(x-2)^3$$



Example. Find the 3rd-degree Taylor polynomial for $f(x) = \arctan(x)$ centered at $a = 0$.

① $P_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$

② $f(x) = \arctan(x)$

$f'(x) = \frac{1}{1+x^2} = (1+x^2)^{-1}$

$f''(x) = -(1+x^2)^{-2} \cdot 2x = \frac{-2x}{(1+x^2)^2}$

$f'''(x) = \frac{(1+x^2)^2 \cdot (-2) - (-2x) \cdot 2(1+x^2) \cdot 2x}{(1+x^2)^4} = \frac{-2(1+x^2) + 8x^2}{(1+x^2)^3} = \frac{6x^2 - 2}{(1+x^2)^3}$

⇓

$f(0) = 0$

$f'(0) = \frac{1}{1+0} = 1$

$f''(0) = \frac{-2 \cdot 0}{(1+0)^2} = 0$

$f'''(0) = \frac{6 \cdot 0 - 2}{(1+0)^3} = -2$

③ $P_3(x) = 0 + x + \frac{0}{2!}x^2 - \frac{2}{3!}x^3$

$$P_3(x) = x - \frac{1}{3}x^3$$

Example. Find the 3rd-degree Taylor polynomial for $f(x) = \ln(1 + x)$ centered at $a = 1$.