11.10 Taylor and Maclaurin Series

Introduction

Taylor polynomials approximate functions locally by matching the value and derivatives of a function at a fixed point. While these polynomials are useful for short-range approximation, they do not always capture the full behavior of a function. In this section, we explore Taylor series, which extend Taylor polynomials to infinite degree. These series allow us to represent functions globally (within a radius of convergence) using an infinite sum of derivatives.

Definition. If a function f has derivatives of all orders at a point a, its **Taylor series centered at** a is:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

The Taylor series centered at $a = 0$ is called the **Maclaurin series**:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

Common Maclaurin Series

The following are standard Maclaurin series for important functions. You are expected to memorize these and be able to use them to generate new series by substitution, differentiation, or integration.

Function	Maclaurin Series	Radius of Convergence
e^x	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	∞
$\sin x$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$	∞
$\cos x$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$	∞
$\frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n$	1
$\ln(1+x)$	$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$	1
$\arctan x$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$	1

Example. Suppose $f^{(n)}(0) = n! \cdot 3^n$ for $n = 0, 1, 2, \ldots$. Find the Maclaurin series for $f(x)$ and determine its radius of convergence.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{n! \cdot 3^n}{n!} x^n = \sum_{n=0}^{\infty} 3^n \cdot x^n$$

$\swarrow (3x)^n$

This is a geometric series that converges when $|3x| < 1 \Rightarrow |x| < \frac{1}{3}$

$$\boxed{R = \frac{1}{3}}$$

Example. Find the Taylor series for $f(x) = \dfrac{1}{x}$ centered at $a = 1$.

- We could compute derivatives and use the Taylor Series formula
- Faster to recognize this as a geometric series

Key idea: If a function can be written as a power series centered at a, and the series converges to the function, then it <u>is</u> the Taylor Series.

$$\frac{1}{x} = \frac{1}{1-(1-x)} = \frac{1}{1-(-(x-1))} = \sum_{n=0}^{\infty} \left(-(x-1)\right)^n = \sum_{n=0}^{\infty} (-1)^n (x-1)^n$$

This converges when $|-(x-1)| < 1 \Rightarrow |x-1| < 1$

Example. Find the Maclaurin series for $f(x) = \arctan(x^2)$.

From the table, we know $\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ for $|x| < 1$

This implies $\arctan(x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{2n+1}$ for $|x^2| < 1$

$\Rightarrow |x| < 1$

Example. Find the sum of the series:

$$\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{4^{2n}(2n)!}$$

Recall: $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ for all x

Hence $\cos\left(\frac{\pi}{4}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{4}\right)^{2n}}{(2n)!} = \frac{\sqrt{2}}{2}$