11.10 Taylor and Maclaurin Polynomials

Taylor polynomials approximate functions locally using finite-degree polynomials. They simplify functions and approximate their behavior near a point. They match a function's value and derivatives at a given point, making them invaluable in physics, engineering, and numerical computation.

Definition. The **Taylor polynomial** of degree n for a function $f(x)$ centered at a is given by:

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

If the Taylor polynomial is centered at $a = 0$, it is called a **Maclaurin polynomial**:

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n.$$

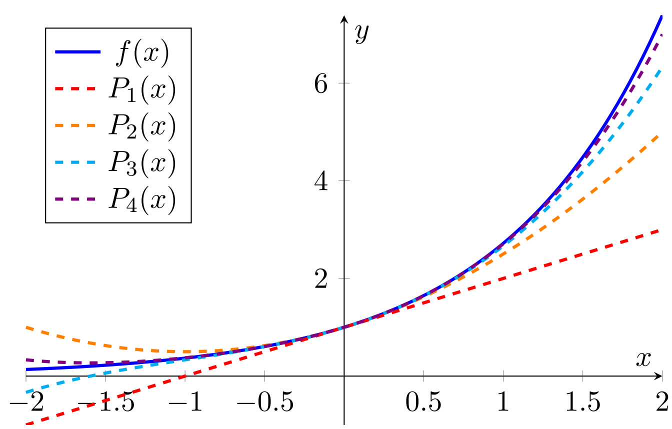
Example. Prove that the nth degree Taylor polynomial $P_n(x)$ of a function $f(x)$, centered at a, has the same value and the same derivatives, up to order n, as $f(x)$ at $x = a$.

- Step 1: Compute the k-th derivative of the Taylor polynomial.

- Step 2: Evaluate the k-th derivative at $x = a$.

- Step 3: Verify the equality for all derivatives up to n.

Example. Compute the 4th-degree Taylor polynomial for $f(x) = e^x$ centered at $a = 0$. Graph $f(x) = e^x$ and its 1st-, 2nd- and 3rd-degree Taylor polynomials to observe local accuracy near $a = 0$.



Steps to Find a Taylor Polynomial

1. Determine the center a of the polynomial and write down the formula

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

2. Compute the derivatives $f'(x), f''(x), \ldots, f^{(n)}(x)$.

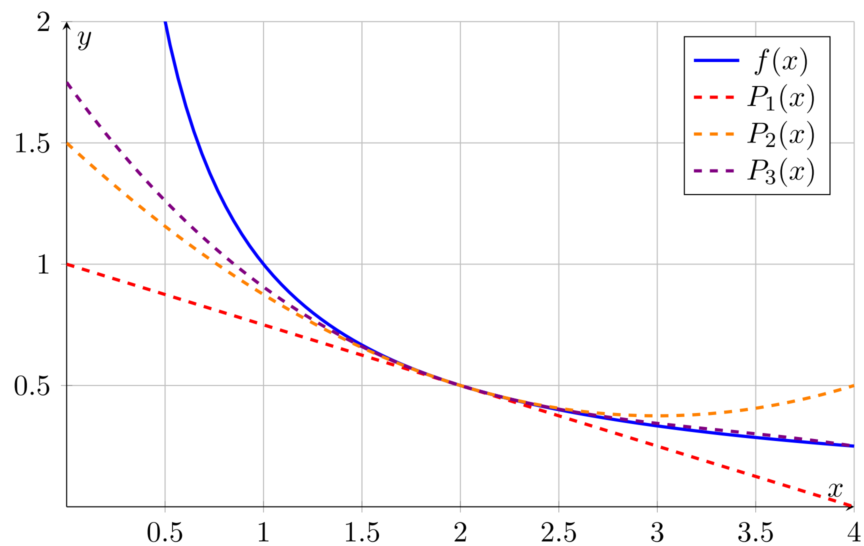
3. Evaluate the function and its derivatives at $x = a$:

$$f(a),\ f'(a),\ f''(a), \ldots,\ f^{(n)}(a).$$

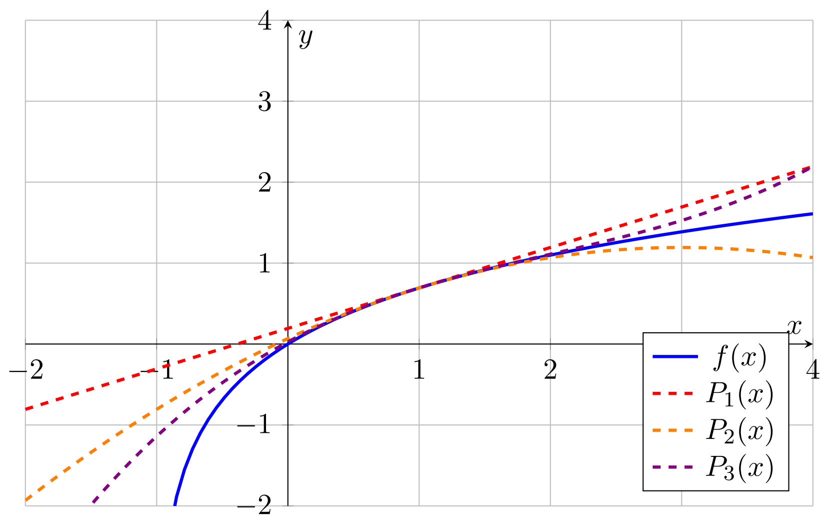
4. Plug these values into the formula for $P_n(x)$.

Example. Find the 3rd-degree Taylor polynomial for $f(x) = \sin(x)$, centered at $a = 0$.

Example. Find the 3rd-degree Taylor polynomial for $f(x) = \frac{1}{x}$, centered at $a = 2$.



Example. Find the Taylor polynomial of degree 3 for $f(x) = \ln(1 + x)$ centered at $a = 1$.



Example. Find the Taylor polynomial of degree 3 for $f(x) = \arctan(x)$ centered at $a = 0$.