11.10 Taylor and Maclaurin Polynomials

Taylor polynomials approximate functions locally using finite-degree polynomials. They simplify functions and approximate their behavior near a point. They match a function's value and derivatives at a given point, making them invaluable in physics, engineering, and numerical computation.

Definition. The **Taylor polynomial** of degree n for a function $f(x)$ centered at a is given by:

← n^{th} derivative at a

$$\frac{f^{(k)}(a)}{k!}(x-a)^k$$

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

If the Taylor polynomial is centered at $a = 0$, it is called a **Maclaurin polynomial**:

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n.$$

Example. Prove that the nth degree Taylor polynomial $P_n(x)$ of a function $f(x)$, centered at a, has the same value and the same derivatives, up to order n, as $f(x)$ at $x = a$.

- Step 1: Compute the k-th derivative of the Taylor polynomial.

$$P_n^{(k)}(x) = \frac{f^{(k)}(a)}{k!} \cdot k! + \text{"terms with } (x-a) \text{ in them still"}$$

this comes from repeatedly using the power rule

- Step 2: Evaluate the k-th derivative at $x = a$.

At $x=a$, all terms involving $(x-a)$ vanish and we're left with

$$P_n^{(k)}(a) = f^{(k)}(a)$$

- Step 3: Verify the equality for all derivatives up to n.

This process holds for all $k \leq n$, so $\boxed{P_n^{(k)}(a) = f^{(k)}(a) \text{ for } k \leq n}$

☆ At a, the derivatives of the Taylor Polynomial are the same as the derivatives of f

Example. Compute the 4th-degree Taylor polynomial for $f(x) = e^x$ centered at $a = 0$. Graph $f(x) = e^x$ and its 1st-, 2nd- and 3rd-degree Taylor polynomials to observe local accuracy near $a = 0$.

① $P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$

② $f(x) = e^x$

$f'(x) = e^x$

$f''(x) = e^x$

$f'''(x) = e^x$

$f^{(4)}(x) = e^x$

③ $f(0) = 1$

$f'(0) = 1$

$f''(0) = 1$

$f'''(0) = 1$

$f^{(4)}(0) = 1$

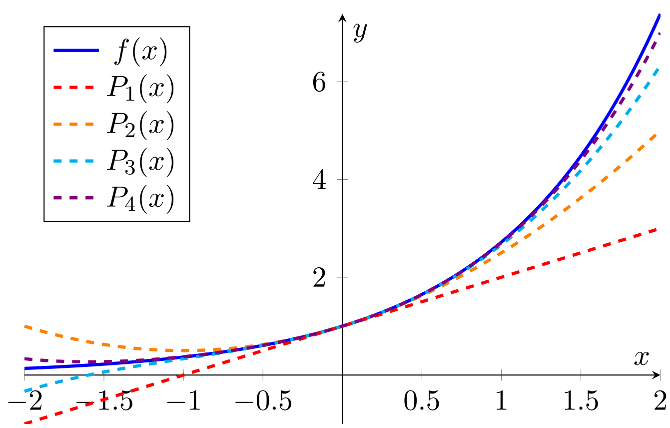
④ $P_4(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$

$P_4(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4$

$P_1(x) = 1 + x$

$P_2(x) = 1 + x + \frac{1}{2}x^2$

$P_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$



Steps to Find a Taylor Polynomial

1. Determine the center a of the polynomial and write down the formula

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

2. Compute the derivatives $f'(x), f''(x), \ldots, f^{(n)}(x)$.
3. Evaluate the function and its derivatives at $x = a$:

$$f(a),\ f'(a),\ f''(a), \ldots,\ f^{(n)}(a).$$

4. Plug these values into the formula for $P_n(x)$.

Example. Find the 3rd-degree Taylor polynomial for $f(x) = \sin(x)$, centered at $a = 0$.

1 $$P_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$$

2
$$f(x) = \sin x$$
$$f'(x) = \cos x$$
$$f''(x) = -\sin x$$
$$f'''(x) = -\cos x$$

3
$$f(0) = \sin(0) = 0$$
$$f'(0) = \cos(0) = 1$$
$$f''(0) = -\sin(0) = 0$$
$$f'''(0) = -\cos(0) = -1$$

4 $$P_3(x) = 0 + x + \frac{0}{2!}x^2 - \frac{1}{3!}x^3$$

$$P_3(x) = x - \frac{x^3}{6}$$

Example. Find the 3rd-degree Taylor polynomial for $f(x) = \frac{1}{x}$, centered at $a = 2$.

① $P_3(x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3$

② $f(x) = \frac{1}{x} = x^{-1}$

$f'(x) = -x^{-2} = -\frac{1}{x^2}$

$f''(x) = 2x^{-3} = \frac{2}{x^3}$

$f'''(x) = -6x^{-4} = \frac{-6}{x^4}$

③ $f(2) = \frac{1}{2}$

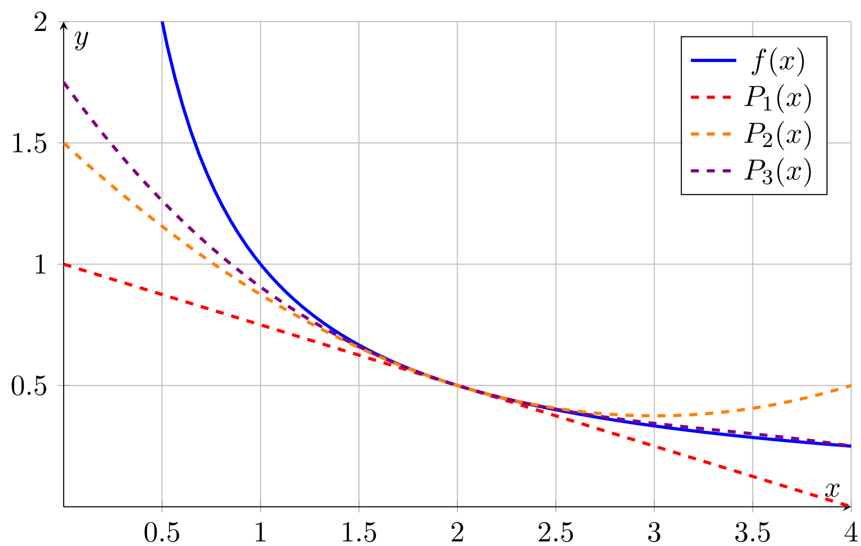
$f'(2) = -\frac{1}{4}$

$f''(2) = \frac{2}{8} = \frac{1}{4}$

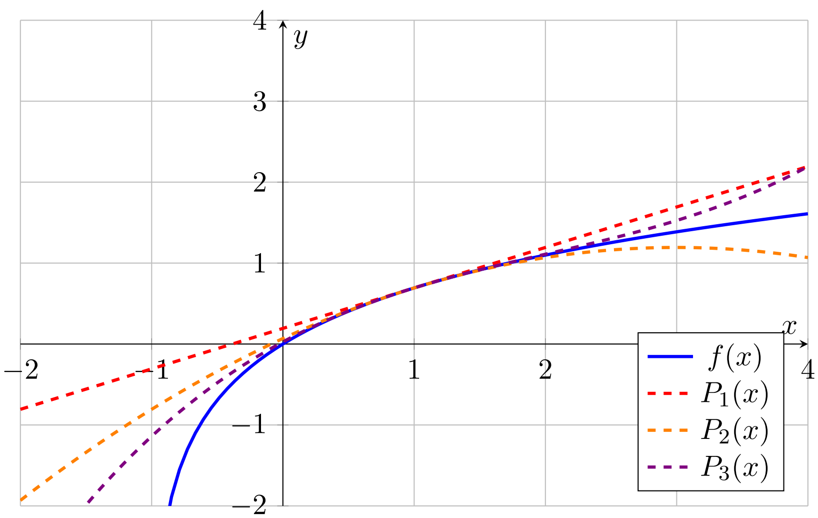
$f'''(2) = \frac{-6}{16} = \frac{-3}{8}$

④ $P_3(x) = \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1/4}{2!}(x-2)^2 - \frac{3/8}{3!}(x-2)^3$

$P_3(x) = \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{8}(x-2)^2 - \frac{1}{16}(x-2)^3$



Example. Find the Taylor polynomial of degree 3 for $f(x) = \ln(1 + x)$ centered at $a = 1$.



Example. Find the Taylor polynomial of degree 3 for $f(x) = \arctan(x)$ centered at $a = 0$.

(1) $P_3(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$

(2) $f(x) = \arctan x$

$f'(x) = \frac{1}{1+x^2} = (1+x^2)^{-1}$

$f''(x) = -(1+x^2)^{-2}\cdot 2x = \frac{-2x}{(1+x^2)^2}$

$f'''(x) = \frac{(1+x^2)^2\cdot(-2) - (-2x)\cdot 2\cdot(1+x^2)\cdot 2x}{(1+x^2)^4} = \frac{-2(1+x^2)+8x^2}{(1+x^2)^3} = \frac{6x^2-2}{(1+x^2)^3}$

(3) $f(0) = \arctan(0) = 0$

$f'(0) = \frac{1}{1+0^2} = 1$

$f''(0) = \frac{-2\cdot 0}{(1+0)^2} = 0$

$f'''(0) = \frac{6\cdot 0^2 - 2}{(1+0)^3} = -2$

(4) $P_3(x) = 0 + 1x + \frac{0}{2!}x^2 + \frac{-2}{3!}x^3$

$P_3(x) = x - \frac{1}{3}x^3$