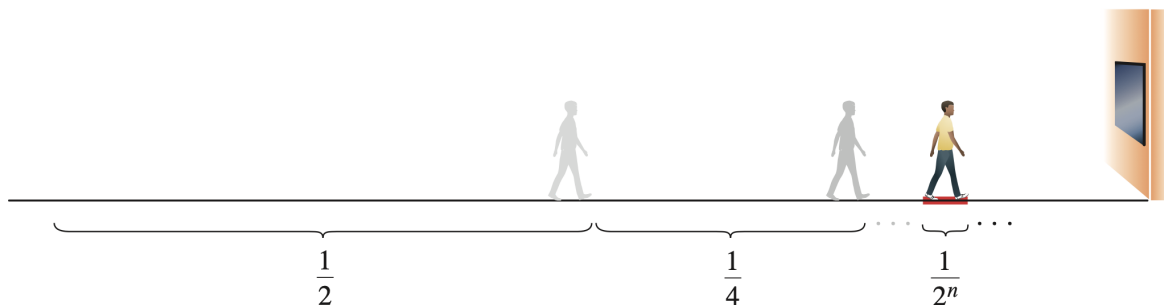


11.1 Sequences

Question. Is it possible for a person standing in a room to walk to a wall?



Definition (Infinite Sequence). An **infinite sequence**, or just a **sequence**, can be thought of as a list of numbers written in a definite order:

$$a_1, a_2, a_3, a_4, \dots, a_n, \dots$$

The number a_1 is called the first term, a_2 is the second term, and in general a_n is the n th term. Each term a_n has a successor a_{n+1} .

Notice that for every positive integer n , there is a corresponding number a_n , so a sequence can be defined as a function f whose domain is the set of positive integers. Typically, we write a_n instead of $f(n)$ to denote the value of the function at n .

Notation: The sequence $\{a_1, a_2, a_3, \dots\}$ is also denoted by:

$$\{a_n\} \quad \text{or} \quad \{a_n\}_{n=1}^{\infty}.$$

Unless otherwise stated, it is assumed that n starts at 1.

Example. The sequence of distances walked by a man in Zeno's paradox can be described as:

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \dots, \frac{1}{2^n}, \dots$$

What are three equivalent descriptions of this sequence?

Example. Write the first few terms of the sequence

$$\left\{ \frac{n}{n+1} \right\}_{n=2}^{\infty}$$

Example. Rewrite the sequence $\{\sqrt{n+2}\}_{n=1}^{\infty}$ in an equivalent way, but start with $n = 3$.

Example. Write the first few terms of the sequence

$$\left\{ (-1)^n \frac{n+1}{3^n} \right\}_{n=0}^{\infty}$$

Example. Find a formula for the general term a_n of the sequence:

$$\left\{ \frac{3}{5}, -\frac{4}{25}, \frac{5}{125}, -\frac{6}{625}, \frac{7}{3125}, \dots \right\}$$

assuming that the pattern of the first few terms continues.

Example. Give an example of a sequence that doesn't have a simple defining equation.

The Limit of a Sequence

Example. Represent the sequence

$$\left\{ \frac{n}{n+1} = \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots \right\}$$

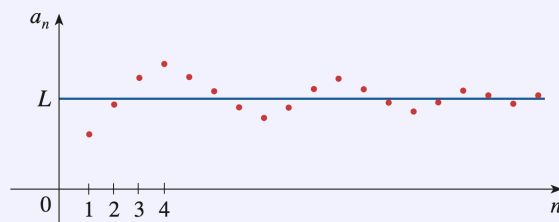
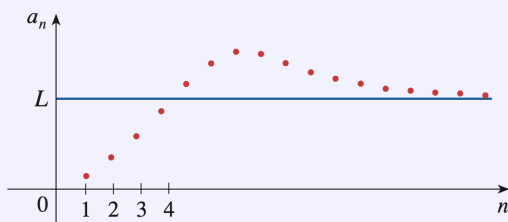
graphically. What happens as n becomes large?

Definition (Intuitive Definition of a Limit of a Sequence). A sequence $\{a_n\}$ has the **limit** L , and we write:

$$\lim_{n \rightarrow \infty} a_n = L \quad \text{or} \quad a_n \rightarrow L \quad \text{as} \quad n \rightarrow \infty,$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large.

If $\lim_{n \rightarrow \infty} a_n$ exists, the sequence is said to converge (or be convergent). Otherwise, the sequence diverges (or is divergent).

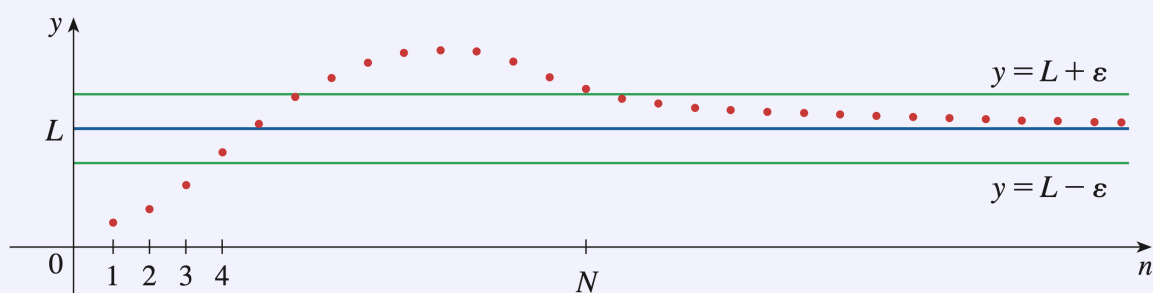


Definition (Precise Definition of a Limit of a Sequence). A sequence $\{a_n\}$ has the **limit** L , and we write:

$$\lim_{n \rightarrow \infty} a_n = L \quad \text{or} \quad a_n \rightarrow L \quad \text{as} \quad n \rightarrow \infty,$$

if for every $\varepsilon > 0$, there is a corresponding integer N such that:

$$\text{if } n > N, \quad \text{then } |a_n - L| < \varepsilon.$$

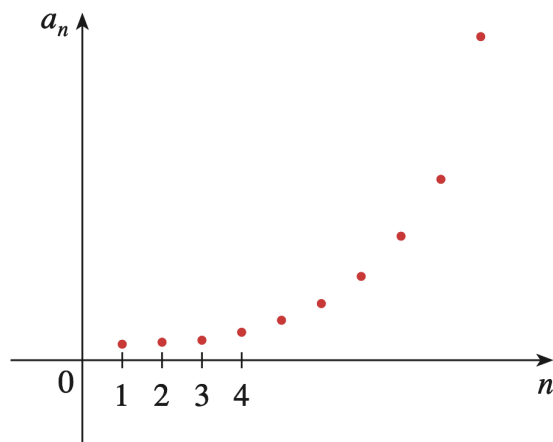
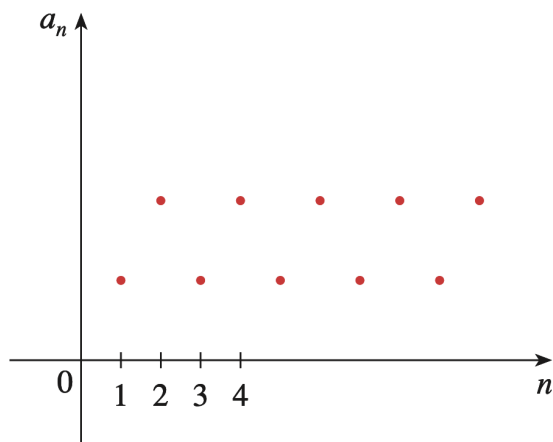


Definition. A sequence **diverges** if its terms do not approach a single number. The notation

$$\lim_{n \rightarrow \infty} a_n = \infty$$

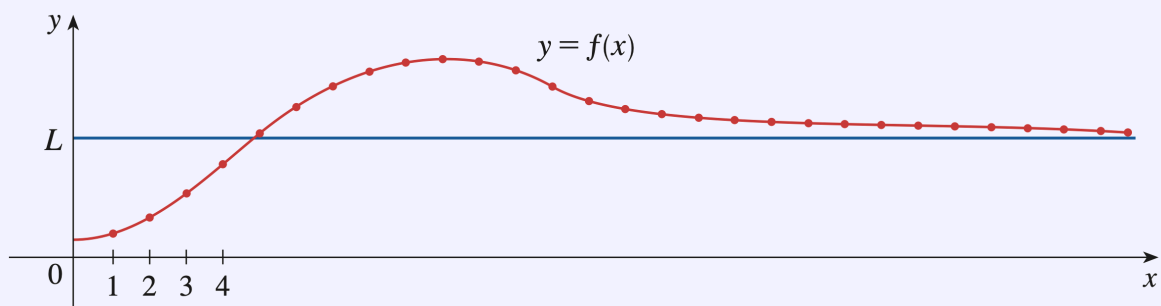
means that for every positive number M there is an integer N such that if $n > N$ then $a_n > M$.

Example. Explain why the following sequences are divergent.



Properties of Convergent Sequences

Theorem. If $\lim_{x \rightarrow \infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim_{n \rightarrow \infty} a_n = L$



Example. Show that $\lim_{n \rightarrow \infty} \frac{1}{n^r} = 0$ if $r > 0$.

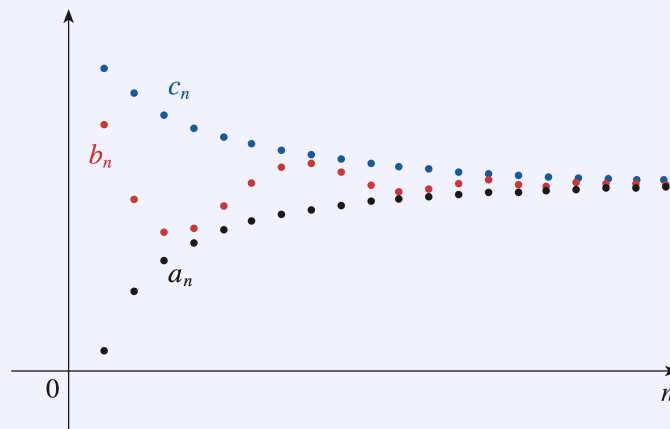
Limit Laws for Sequences. Let $\{a_n\}$ and $\{b_n\}$ be convergent sequences, and let c be a constant. The following limit laws hold:

Sum Law	$\lim_{n \rightarrow \infty} (a_n + b_n) = \lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n$
Difference Law	$\lim_{n \rightarrow \infty} (a_n - b_n) = \lim_{n \rightarrow \infty} a_n - \lim_{n \rightarrow \infty} b_n$
Constant Multiple Law	$\lim_{n \rightarrow \infty} (ca_n) = c \cdot \lim_{n \rightarrow \infty} a_n$
Product Law	$\lim_{n \rightarrow \infty} (a_n b_n) = \lim_{n \rightarrow \infty} a_n \cdot \lim_{n \rightarrow \infty} b_n$
Quotient Law	$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{\lim_{n \rightarrow \infty} a_n}{\lim_{n \rightarrow \infty} b_n}, \quad \text{provided } \lim_{n \rightarrow \infty} b_n \neq 0.$

Power Law for Sequences. If $\lim_{n \rightarrow \infty} a_n = A$, $a_n > 0$ and $p > 0$, then:

$$\lim_{n \rightarrow \infty} (a_n^p) = \left[\lim_{n \rightarrow \infty} a_n \right]^p = A^p.$$

Squeeze Theorem for Sequences. Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences such that $a_n \leq b_n \leq c_n$ for all $n > n_0$ (for some n_0). If $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = L$, then $\lim_{n \rightarrow \infty} b_n = L$.



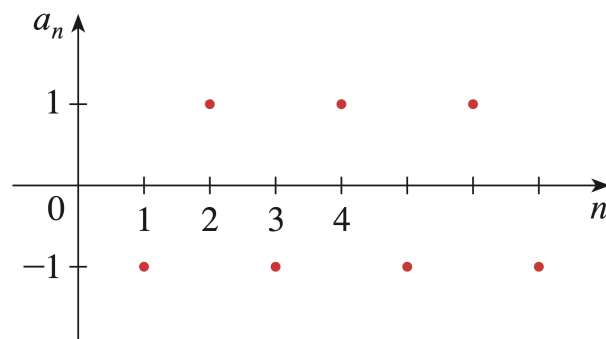
Theorem. If $\lim_{n \rightarrow \infty} |a_n| = 0$, then $\lim_{n \rightarrow \infty} a_n = 0$.

Example. Find $\lim_{n \rightarrow \infty} \frac{n}{n+1}$.

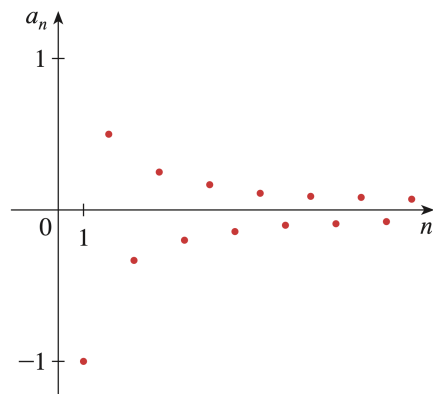
Example. Determine whether the sequence $a_n = \frac{n}{\sqrt{10+n}}$ is convergent or divergent.

Example. Calculate $\lim_{n \rightarrow \infty} \frac{\ln n}{n}$.

Example. Determine whether the sequence $a_n = (-1)^n$ is convergent or divergent.



Example. Evaluate $\lim_{n \rightarrow \infty} \frac{(-1)^n}{n}$ if it exists.



Theorem. If $\lim_{n \rightarrow \infty} a_n = L$ and the function f is continuous at L , then:

$$\lim_{n \rightarrow \infty} f(a_n) = f(L).$$

Example. Find $\lim_{n \rightarrow \infty} \sin\left(\frac{1}{n}\right)$.

Monotonic and Bounded Sequences

Definition. A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \geq 1$, that is, $a_1 < a_2 < a_3 < \cdots$. It is called **decreasing** if $a_n > a_{n+1}$ for all $n \geq 1$. A sequence is called **monotonic** if it is either increasing or decreasing.

Example. Show that the sequence $a_n = \frac{3}{n+5}$ is decreasing.

Example. Show that the sequence $a_n = \frac{n}{n^2+1}$ is decreasing.

Definition. A sequence $\{a_n\}$ is **bounded above** if there is a number M such that:

$$a_n \leq M \quad \text{for all } n \geq 1.$$

A sequence is **bounded below** if there is a number m such that:

$$m \leq a_n \quad \text{for all } n \geq 1.$$

If a sequence is bounded above and below, then it is called a **bounded sequence**.

Example. Give an example of a sequence that is bounded below but not above.

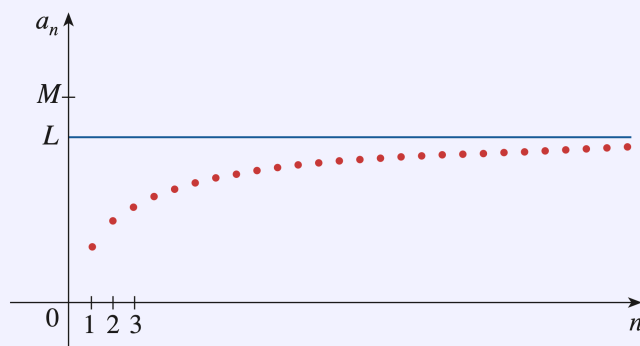
Example. Give an example of a bounded sequence.

Example. True or False: every bounded sequence is convergent.

Example. True or False: every monotonic sequence is convergent.

Example. True or False: every bounded, monotonic sequence is convergent.

Theorem. Every bounded, monotonic sequence is convergent.



In other words:

- A sequence that is increasing and bounded above converges.
- A sequence that is decreasing and bounded below converges.