11.1 Sequences

Overview

Basic Definitions

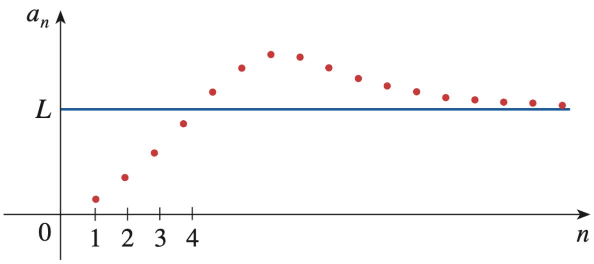
Definition. An *infinite sequence* is a function $a : \mathbb{N} \to \mathbb{R}$, usually written as $a_1, a_2, a_3, \ldots$ or equivalently as $\{a_n\}_{n=1}^{\infty}$.

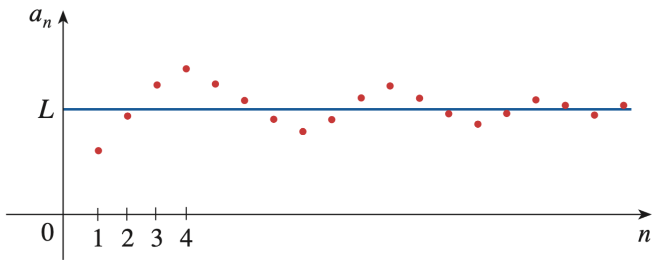
Definition. A sequence $\{a_n\}$ has the **limit** L, and we write:

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \quad \text{as} \quad n \to \infty,$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large.

If $\lim_{n\to\infty} a_n$ exists, the sequence is said to converge (or be convergent). Otherwise, the sequence diverges (or is divergent).





Limit Laws for Sequences

Suppose $\{a_n\}$ and $\{b_n\}$ are convergent sequences and let c be a constant. Then:

1. **Sum Law:**
$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n.$$
2. **Difference Law:**
$$\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n.$$
3. **Constant Multiple Law:**
$$\lim_{n\to\infty} (c\, a_n) = c \lim_{n\to\infty} a_n.$$
4. **Product Law:**
$$\lim_{n\to\infty} (a_n b_n) = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right).$$
5. **Quotient Law:** If $\lim_{n\to\infty} b_n \neq 0$, then
$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}.$$
6. **Power Law:** For any real number $p > 0$ (and with $a_n > 0$),
$$\lim_{n\to\infty} (a_n^p) = \left(\lim_{n\to\infty} a_n\right)^p.$$

Convergence Theorems

Theorem. If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim_{n\to\infty} a_n = L$

Theorem (Squeeze Theorem for Sequences)**.** If $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ are sequences satisfying $a_n \le b_n \le c_n$ for all $n \ge N$, and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

Theorem (Absolute Value Theorem)**.** If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Theorem (Limits and Continuity)**.** If $\lim_{n\to\infty} a_n = L$ and the function f is continuous at L, then $\lim_{n\to\infty} f(a_n) = f(L)$.

Theorem (Geometric Sequences)**.** Consider the sequence $\{r^n\}$.

- If $-1 < r < 1$, then $\lim_{n\to\infty} r^n = 0$.
- If $r = 1$, then $\lim_{n\to\infty} r^n = 1$.
- If $r \le -1$ or $r > 1$, the sequence $\{r^n\}$ diverges (in the case $r \le -1$, the terms oscillate without approaching a single value).

Theorem (Monotonic Sequence Theorem)**.** Every bounded, monotonic sequence converges.
A sequence is said to be:

- **Increasing** if $a_{n+1} \ge a_n$ for all n.
- **Decreasing** if $a_{n+1} \le a_n$ for all n.
- **Monotonic** if it is either increasing or decreasing.

A sequence $\{a_n\}$ is *bounded above* if there exists a number M such that $a_n \le M$ for all n, and *bounded below* if there exists m such that $a_n \ge m$ for all n. If both conditions hold, the sequence is called *bounded.*

Sequences Problems

1. Determine whether the sequence $a_n = \dfrac{n}{n+1}$ converges or diverges. If it converges, find its limit.

2. Determine whether the sequence $b_n = (-1)^n$ converges or diverges. If it converges, find its limit.

3. Determine whether the sequence $c_n = \dfrac{1}{n^2}$ converges or diverges. If it converges, find its limit.

4. Determine whether the sequence $a_n = \dfrac{n^2}{n^2+1}$. converges or diverges. If it converges, find its limit.

5. Determine whether the sequence $b_n = \dfrac{\ln n}{n}$ converges or diverges. If it converges, find its limit.

6. Determine whether the sequence $c_n = \dfrac{n}{\sqrt{n^2+1}}$ converges or diverges. If it converges, find its limit.

7. Determine whether the sequence $d_n = \dfrac{(-1)^n}{n}$ converges or diverges. If it converges, find its limit.

8. Determine whether the sequence $a_n = \dfrac{\cos(3n+1)}{n^2}$ converges or diverges. If it converges, find its limit.

9. Determine whether the sequence $c_n = \dfrac{n^2-3n}{n^3+5}$ converges or diverges. If it converges, find its limit.

10. Determine whether the sequence $a_n = \dfrac{\sqrt{9n^5+4n^2}}{n^3}$ converges or diverges. If it converges, find its limit.

11. Determine whether the sequence $b_n = \dfrac{5n+\sin(n)}{n+10}$. converges or diverges. If it converges, find its limit.

12. Determine whether the sequence $c_n = \dfrac{n^3+2}{\sqrt{n^6+5n^2}}$ converge or diverge? If it converges, find its limit.

13. Determine whether the sequence $a_n = \dfrac{\sin(5n)}{n^3}$ converges or diverges. If it converges, find its limit.

14. Determine whether the sequence $b_n = \dfrac{3\sqrt{n}+n^3}{n^3+\sqrt{n}}$ converges or diverges. If it converges, find its limit.

15. Determine whether the sequence $c_n = \dfrac{n^3-2n}{\sqrt{4n^6+7n}}$ converges or diverges. If it converges, find its limit.