11.1 Sequences (Solutions)

1. We rewrite:

$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}}.$$

   Since $\frac{1}{n} \to 0$ as $n \to \infty$, we get:

$$\lim_{n\to\infty} \frac{n}{n+1} = 1.$$

   Thus, the sequence converges to 1.

2. The sequence oscillates between 1 and -1 and does not settle to a single value. Since it does not approach a single limit, the sequence diverges.

3. Since $\frac{1}{n^2} > 0$ for all n and $\frac{1}{n^2} \to 0$ as $n \to \infty$, the sequence converges to 0.

4. Dividing the numerator and denominator by n^2, we get:

$$\lim_{n\to\infty} \frac{n^2}{n^2+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n^2}}.$$

   Since $\frac{1}{n^2} \to 0$, the limit is 1. Thus, the sequence converges to 1.

5. Applying L'Hôpital's Rule to the related function $f(x) = \frac{\ln x}{x}$, we differentiate:

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = \lim_{x\to\infty} \frac{1}{x} = 0.$$

   Thus, $\lim_{n\to\infty} \frac{\ln n}{n} = 0$, meaning the sequence converges to 0.

6. Dividing by n:

$$\lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n^2}}}.$$

   Since $\frac{1}{n^2} \to 0$, we get:

$$\lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = \frac{1}{\sqrt{1}} = 1.$$

   Thus, the sequence converges to 1.

7. Since $|d_n| = \frac{1}{n} \to 0$, d_n converges to 0 as well.

8. Since $-1 \le \cos(3n+1) \le 1$, we have

$$-\frac{1}{n^2} \le \frac{\cos(3n+1)}{n^2} \le \frac{1}{n^2}.$$

   Since $\frac{1}{n^2} \to 0$ as $n \to \infty$, by the Squeeze Theorem, the sequence converges to 0.

9. Dividing numerator and denominator by n^3:

$$c_n = \frac{\frac{n^2}{n^3} - \frac{3n}{n^3}}{\frac{n^3}{n^3} + \frac{5}{n^3}} = \frac{\frac{1}{n} - \frac{3}{n^2}}{1 + \frac{5}{n^3}}.$$

Taking the limit as $n \to \infty$:

$$\lim_{n\to\infty} c_n = \frac{0 - 0}{1 + 0} = 0.$$

Thus, the sequence converges to 0.

10. Factor out n^4 inside the square root:

$$\sqrt{9n^4 + 4n^2} = \sqrt{n^4\left(9 + \frac{4}{n^2}\right)} = n^2\sqrt{9 + \frac{4}{n^2}}.$$

Thus,

$$a_n = \frac{n^2\sqrt{9 + \frac{4}{n^2}}}{n^3} = \frac{\sqrt{9 + \frac{4}{n^2}}}{n}.$$

As $n \to \infty$, the term $\sqrt{9 + \frac{4}{n^2}} \to 3$. Hence,

$$a_n \to \frac{3}{n} \to 0.$$

Therefore, the sequence $\{a_n\}$ converges and its limit is 0.

11. Dividing numerator and denominator by n:

$$b_n = \frac{5 + \frac{\sin(n)}{n}}{1 + \frac{10}{n}}.$$

Since $\frac{\sin(n)}{n} \to 0$ and $\frac{10}{n} \to 0$, we get:

$$\lim_{n\to\infty} b_n = \frac{5 + 0}{1 + 0} = 5.$$

Thus, the sequence converges to 5.

12. Dividing by n^3 in the numerator and n^3 inside the square root in the denominator:

$$c_n = \frac{1 + \frac{2}{n^3}}{\sqrt{1 + \frac{5}{n^4}}}.$$

Taking the limit:

$$\lim_{n\to\infty} c_n = \frac{1 + 0}{\sqrt{1 + 0}} = 1.$$

Thus, the sequence converges to 1.

13. Since $|\sin(5n)| \leq 1$, we have:

$$-\frac{1}{n^3} \leq \frac{\sin(5n)}{n^3} \leq \frac{1}{n^3}.$$

Since $\frac{1}{n^3} \to 0$, by the Squeeze Theorem, $\lim_{n\to\infty} a_n = 0$. The sequence converges to 0.

14. Dividing numerator and denominator by n^3:

$$b_n = \frac{\frac{3\sqrt{n}}{n^3} + 1}{1 + \frac{\sqrt{n}}{n^3}}.$$

Taking the limit as $n \to \infty$, the terms with $\frac{\sqrt{n}}{n^3} \to 0$, so:

$$\lim_{n\to\infty} b_n = \frac{0+1}{1+0} = 1.$$

Thus, the sequence converges to 1.

15. Dividing by n^3:

$$c_n = \frac{1 - \frac{2}{n^2}}{\sqrt{4 + \frac{7}{n^5}}}.$$

Taking the limit:

$$\lim_{n\to\infty} c_n = \frac{1-0}{\sqrt{4+0}} = \frac{1}{2}.$$

Thus, the sequence converges to $\frac{1}{2}$.