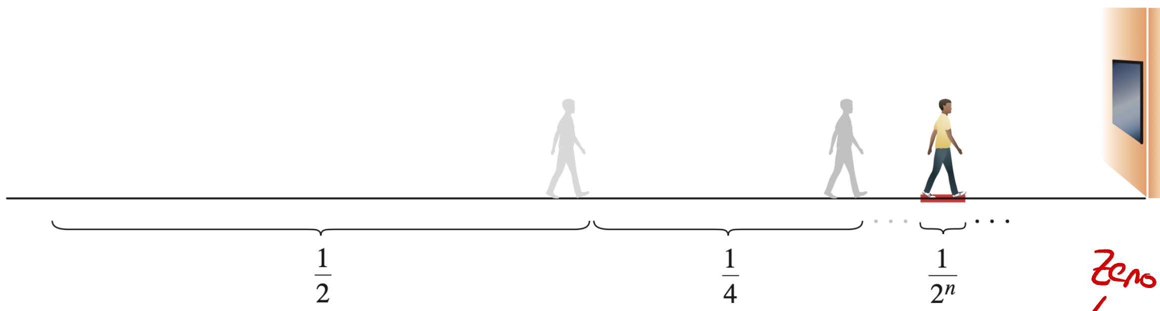
11.1 Sequences

Question. Is it possible for a person standing in a room to walk to a wall?



Zeno's Paradox

The person would first have to walk half the distance to the wall, then half the remaining distance, and so on...

This would require one to complete an infinite number of tasks, which Zeno says is impossible.

Either way, these distances form a sequence: $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots$

Definition (Infinite Sequence)**.** An **infinite sequence**, or just a **sequence**, can be thought of as a list of numbers written in a definite order:

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$$

The number a_1 is called the first term, a_2 is the second term, and in general a_n is the nth term. Each term a_n has a successor a_{n+1}.

Notice that for every positive integer n, there is a corresponding number a_n, so a sequence can be defined as a function f whose domain is the set of positive integers. Typically, we write a_n instead of $f(n)$ to denote the value of the function at n.

Notation: The sequence $\{a_1, a_2, a_3, \ldots\}$ is also denoted by:

$$\{a_n\} \quad \text{or} \quad \{a_n\}_{n=1}^{\infty}.$$

Unless otherwise stated, it is assumed that n starts at 1.

Example. The sequence of distances walked by a man in Zeno's paradox can be described as:

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \dots, \frac{1}{2^n}, \dots$$

What are three equivalent descriptions of this sequence?

$\{a_n\} = \{\frac{1}{2^n}\}$

$\{a_n\} = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots\}$

$a_1 = \frac{1}{2}, \quad a_2 = \frac{1}{4}, \quad a_3 = \frac{1}{8}, \dots,$

Example. Write the first few terms of the sequence

$$\left\{\frac{n}{n+1}\right\}_{n=2}^{\infty}$$

$\{\frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \dots\}$

Example. Rewrite the sequence $\{\sqrt{n+2}\}_{n=1}^{\infty}$ in an equivalent way, but start with $n = 3$.

This is $\sqrt{3}, \sqrt{4}, \sqrt{5}, \dots$

$\{\sqrt{n}\}_{n=3}^{\infty}$ or $a_n = \sqrt{n}$, for $n \geq 3$

Example. Write the first few terms of the sequence

$$\left\{(-1)^n \frac{n+1}{3^n}\right\}_{n=0}^{\infty}$$

$\{1, -\frac{2}{3}, \frac{3}{9}, -\frac{4}{27}, \frac{5}{81}, \dots\}$

Note: The $(-1)^n$ factor creates terms that alternate between positive and negative.

Example. Find a formula for the general term a_n of the sequence:

$$\left\{\frac{3}{5}, -\frac{4}{25}, \frac{5}{125}, -\frac{6}{625}, \frac{7}{3125}, \dots\right\}$$

assuming that the pattern of the first few terms continues.

We are given

$$a_1 = \frac{3}{5}, \quad a_2 = \frac{-4}{25}, \quad a_3 = \frac{5}{125}, \quad a_4 = -\frac{6}{625}, \quad a_5 = \frac{7}{3125}$$

- The numerators increase by 1 each step, starting at 3. $\boxed{n+2}$
- The denominators are powers of 5 $\boxed{5^n}$
- The terms alternate in sign, so we include $(-1)^{n-1}$ or $(-1)^{n+1}$

$$\boxed{a_n = (-1)^{n-1} \frac{n+2}{5^n}}$$

Example. Give an example of a sequence that doesn't have a simple defining equation.

The Fibonacci sequence $\{f_n\}$ is defined recursively by:

$$f_1 = 1, \quad f_2 = 1, \quad f_n = f_{n-1} + f_{n-2} \quad n \geq 3$$

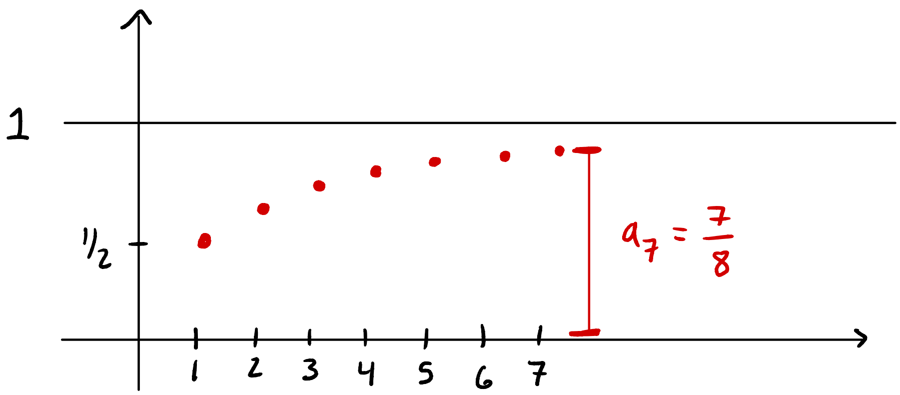
The first few terms are $\{1, 1, 2, 3, 5, 8, 13, 21, \dots\}$

The Limit of a Sequence

Example. Represent the sequence

$$\left\{ \frac{n}{n+1} = \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots \right\}$$

graphically. What happens as n becomes large?



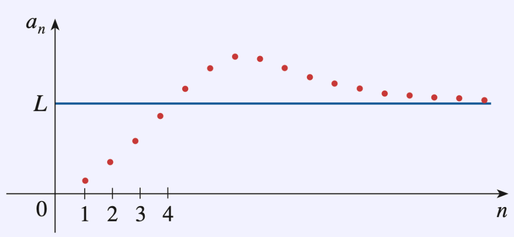
The terms appear to approach 1

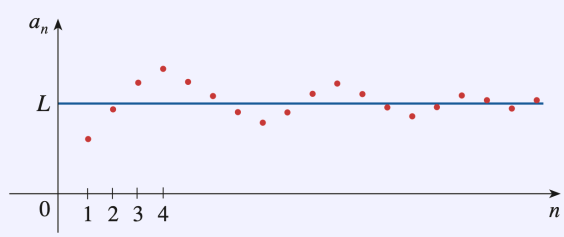
Definition (Intuitive Definition of a Limit of a Sequence)**.** A sequence $\{a_n\}$ has the **limit** L, and we write:

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \quad \text{as} \quad n \to \infty,$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large.

If $\lim_{n\to\infty} a_n$ exists, the sequence is said to converge (or be convergent). Otherwise, the sequence diverges (or is divergent).





This is the definition used in "Analysis"

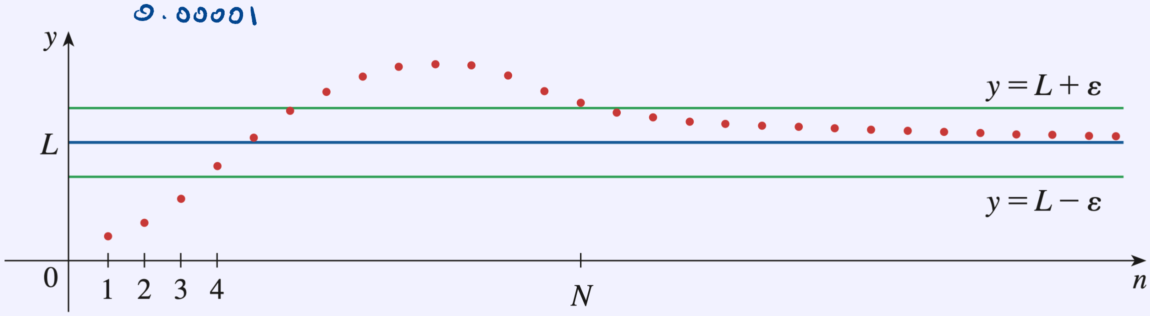
Definition (Precise Definition of a Limit of a Sequence)**.** A sequence $\{a_n\}$ has the **limit** L, and we write:

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \quad \text{as} \quad n \to \infty,$$

if for every $\varepsilon > 0$, there is a corresponding integer N such that:

0.01
0.001
0.00001

$$\text{if } n > N, \quad \text{then } |a_n - L| < \varepsilon.$$

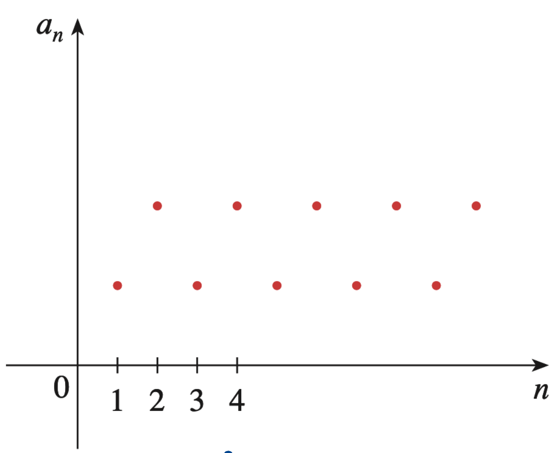


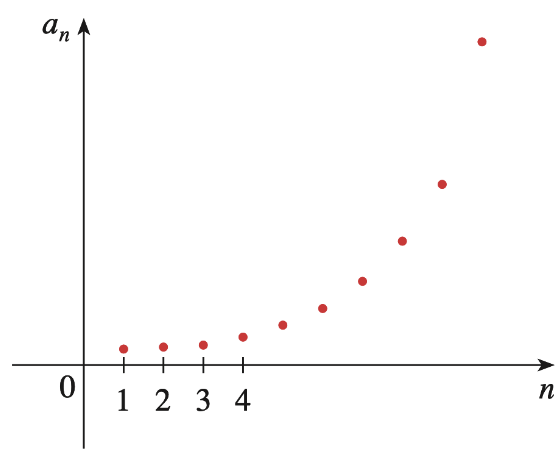
Definition. A sequence **diverges** if its terms do not approach a single number. The notation

$$\lim_{n\to\infty} a_n = \infty$$

means that for every positive number M there is an integer N such that if $n > N$ then $a_n > M$.

Example. Explain why the following sequences are divergent.





Oscillates between values and doesn't approach a single value

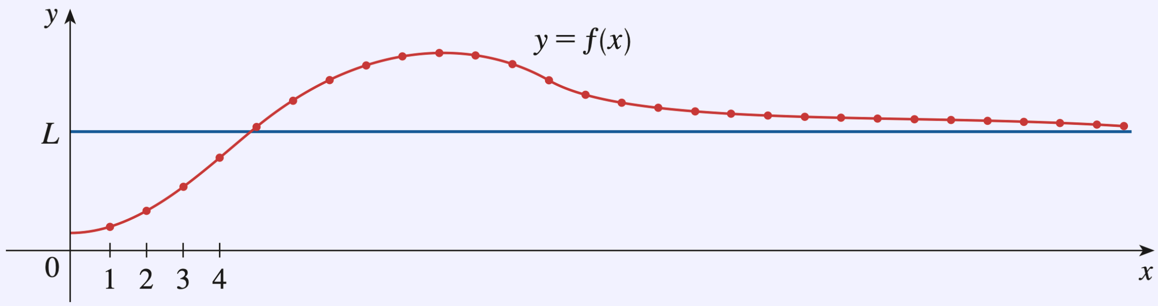
increases without bound

$\lim_{n\to\infty} a_n = \infty$

If f(x) converges, f(n) converges

Properties of Convergent Sequences

Theorem. If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim_{n\to\infty} a_n = L$



Example. Show that $\displaystyle\lim_{n\to\infty} \frac{1}{n^r} = 0$ if $r > 0$.

$\left\{ \frac{1}{1^r}, \frac{1}{2^r}, \frac{1}{3^r}, \frac{1}{4^r}, \dots \right\}$

Let $r>0$

We know $\displaystyle\lim_{x\to\infty} \frac{1}{x^r} = 0$

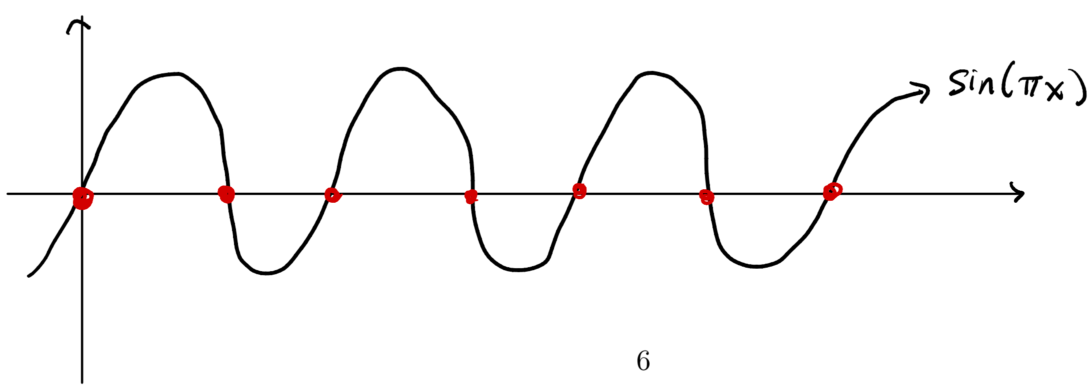
By the theorem, $\displaystyle\lim_{n\to\infty} \frac{1}{n^r} = 0$ as well.

⚠ If $\displaystyle\lim_{x\to\infty} f(x)$ diverges, $\displaystyle\lim_{n\to\infty} f(n)$ may or may not diverge.

Consider $\displaystyle\lim_{n\to\infty} \sin(\pi n)$ vs. $\displaystyle\lim_{x\to\infty} \sin(\pi x)$

$\lim_{n\to\infty} \sin(\pi n) = 0$

↳ diverges by oscillation



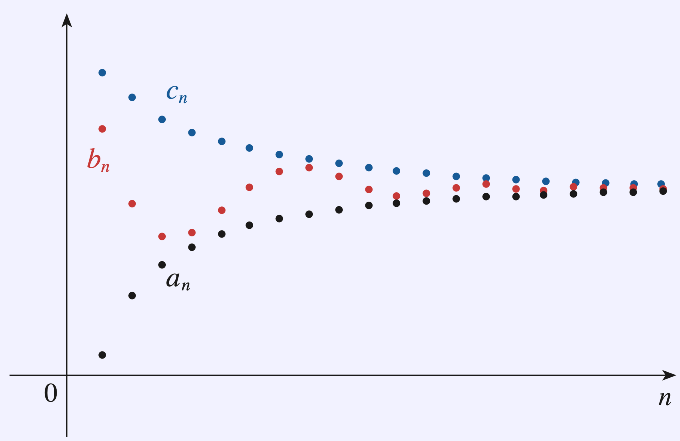
Limit Laws for Sequences. Let $\{a_n\}$ and $\{b_n\}$ be convergent sequences, and let c be a constant. The following limit laws hold:

Sum Law	$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$
Difference Law	$\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$
Constant Multiple Law	$\lim_{n\to\infty}(ca_n) = c \cdot \lim_{n\to\infty} a_n$
Product Law	$\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$
Quotient Law	$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$, provided $\lim_{n\to\infty} b_n \neq 0$.

Power Law for Sequences. If $\lim_{n\to\infty} a_n = A$, $a_n > 0$ and $p > 0$, then:

$$\lim_{n\to\infty}(a_n^p) = \left[\lim_{n\to\infty} a_n\right]^p = A^p.$$

Squeeze Theorem for Sequences. Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences such that $a_n \leq b_n \leq c_n$ for all $n > n_0$ (for some n_0). If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.



Theorem. If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Q: Does $a_n = \frac{\cos(3n+1)}{n^2}$ ✓ converge or diverge?

A: $\frac{-1}{n^2} \leq \frac{\cos(3n+1)}{n^2} \leq \frac{1}{n^2}$

limits as $n\to\infty$ → $\downarrow 0$ → $\downarrow 0$ → $\downarrow 0$

Example. Find $\lim_{n\to\infty} \dfrac{n}{n+1}$.

$$= \lim_{n\to\infty} \frac{n}{n+1} \cdot \frac{1/n}{1/n}$$

$$= \lim_{n\to\infty} \frac{1}{1+1/n}$$

$$= \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)}$$

$$= \frac{1}{\lim_{n\to\infty} 1 + \lim_{n\to\infty} \frac{1}{n}} = \frac{1}{1+0} = 1$$

Example. Determine whether the sequence $a_n = \dfrac{n}{\sqrt{10+n}}$ is convergent or divergent.

$$\lim_{n\to\infty} \frac{n}{\sqrt{10+n}} \cdot \frac{1/n}{1/n} = \lim_{n\to\infty} \frac{1}{\frac{1}{n}\sqrt{10+n}}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{\frac{10}{n^2} + \frac{1}{n}}}$$

$$\frac{1}{n}\sqrt{10+n} = \sqrt{\frac{1}{n^2}} \cdot \sqrt{10+n}$$

$$= \sqrt{\frac{1}{n^2}(10+n)}$$

The numerator is a positive constant.

The denominator is positive and approaches 0.

Hence $\{a_n\}$ is divergent. (It goes to ∞)

Example. Calculate $\lim_{n\to\infty} \frac{\ln n}{n}$. (Assuming n is an integer)

Both the numerator and denominator approach ∞ as $n \to \infty$

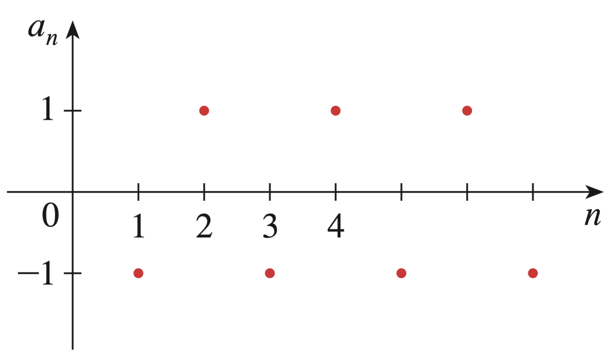
Can't apply L'Hôpital's rule directly because it only applies to functions of a real variable, not sequences.

Apply L'Hôpital's rule to $f(x) = \frac{\ln x}{x}$ instead:

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = \lim_{x\to\infty} \frac{1}{x} = 0$$

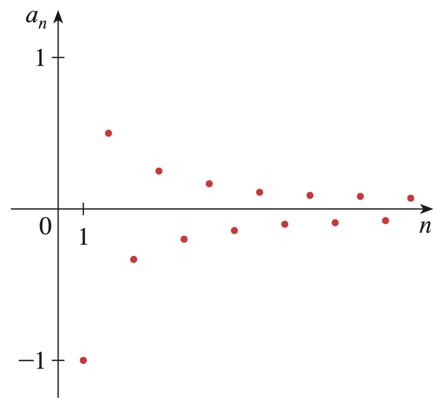
We conclude $\lim_{n\to\infty} \frac{\ln n}{n} = 0$

Example. Determine whether the sequence $a_n = (-1)^n$ is convergent or divergent.



The terms oscillate as $n \to \infty$, so the sequence is divergent.

Example. Evaluate $\lim_{n\to\infty} \frac{(-1)^n}{n}$ if it exists.



Note: $|a_n| = \left|\frac{(-1)^n}{n}\right| = \frac{1}{n}$

As $n \to \infty$, $\frac{1}{n} \to 0$

Since $|a_n| \to 0$, we conclude $a_n \to 0$

So $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$

Theorem. If $\lim_{n\to\infty} a_n = L$ and the function f is continuous at L, then:

$$\lim_{n\to\infty} f(a_n) = f(L).$$

$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right)$

Example. Find $\lim_{n\to\infty} \sin\left(\frac{1}{n}\right)$.

As $n \to \infty$, the terms $\frac{1}{n} \to 0$

$\sin(x)$ is continuous

$\lim_{n\to\infty} \sin\left(\frac{1}{n}\right) = \sin\left(\lim_{n\to\infty} \frac{1}{n}\right) = \sin(0) = 0$

Idea: Monotonic + Bounded = Convergent

Monotonic and Bounded Sequences

Definition. A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \geq 1$, that is, $a_1 < a_2 < a_3 < \cdots$. It is called **decreasing** if $a_n > a_{n+1}$ for all $n \geq 1$. A sequence is called **monotonic** if it is either increasing or decreasing.

Example. Show that the sequence $a_n = \dfrac{3}{n+5}$ is decreasing.

We need to check $a_{n+1} < a_n$ for all $n \geq 1$

$$a_n - a_{n+1} = \frac{3}{n+5} - \frac{3}{n+6} = \frac{3(n+6) - 3(n+5)}{(n+5)(n+6)} = \frac{3}{(n+5)(n+6)}$$

For $n \geq 1$, both $(n+5)$ and $(n+6)$ are positive, $a_n - a_{n+1} > 0$

Hence $a_n > a_{n+1}$ and $\{a_n\}$ is decreasing.

Example. Show that the sequence $a_n = \dfrac{n}{n^2+1}$ is decreasing.

Consider $f(x) = \dfrac{x}{x^2+1}$

Compute the derivative:

$$f'(x) = \frac{(x^2+1)(1) - x(2x)}{(x^2+1)^2} = \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$$

For $x > 1$, $1 - x^2 < 0$. So $f'(x) < 0$ for $x > 1$.

So $f(x)$ is decreasing on $(1, \infty)$

So $f(n) > f(n+1)$ and $\{a_n\}$ is decreasing.

Definition. A sequence $\{a_n\}$ is **bounded above** if there is a number M such that:

$$a_n \leq M \quad \text{for all } n \geq 1.$$

A sequence is **bounded below** if there is a number m such that:

$$m \leq a_n \quad \text{for all } n \geq 1.$$

If a sequence is bounded above and below, then it is called a **bounded sequence**.

Example. Give an example of a sequence that is bounded below but not above.

$a_n = n$ is bounded below by 0, but not bounded above

Example. Give an example of a bounded sequence.

$a_n = \frac{n}{n+1}$ is bounded between 0 and 1

Example. True or False: every bounded sequence is convergent.

False. $(-1)^n$ is bounded but diverges by oscillation.

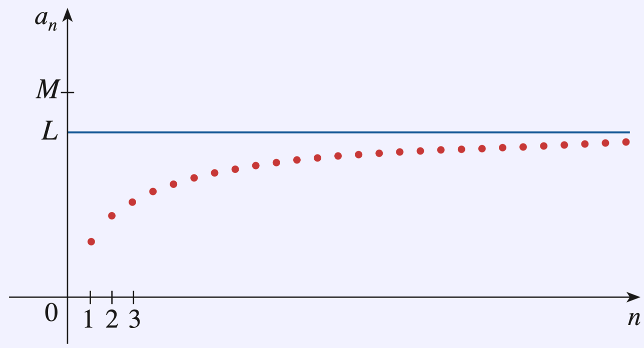
Example. True or False: every monotonic sequence is convergent.

False. $a_n = n$ is monotonic but diverges

Example. True or False: every bounded, monotonic sequence is convergent.

True.

Theorem. Every bounded, monotonic sequence is convergent.



In other words:

- A sequence that is increasing and bounded above converges.
- A sequence that is decreasing and bounded below converges.

The terms are forced to crowd together
and approach some number L.