

10.4 Calculus in Polar Coordinates

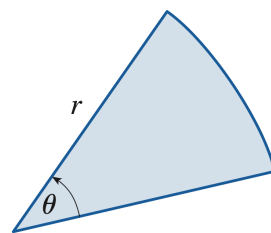
Theorem. Let $r = f(\theta)$ be a continuous function defining a curve in polar coordinates. The area A enclosed by the curve between the angles $\theta = a$ and $\theta = b$ is given by

$$A = \int_a^b \frac{1}{2} r^2 d\theta.$$

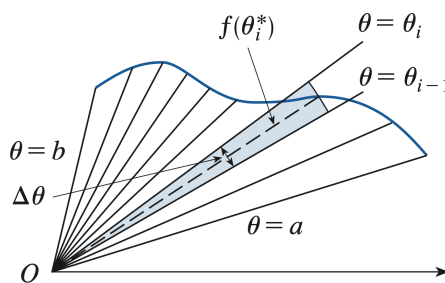
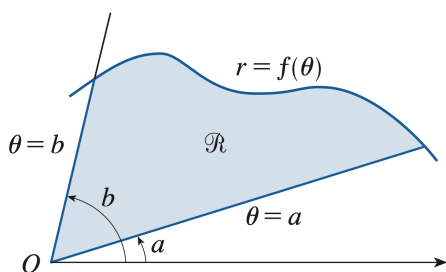
The area is interpreted as being swept out by a rotating ray through the origin O that starts at angle a and ends at angle b .

Proof.

1. What is the area of a sector of a circle?



2. Consider the area enclosed by the polar curve $r = f(\theta)$ between angles $\theta = a$ and $\theta = b$.

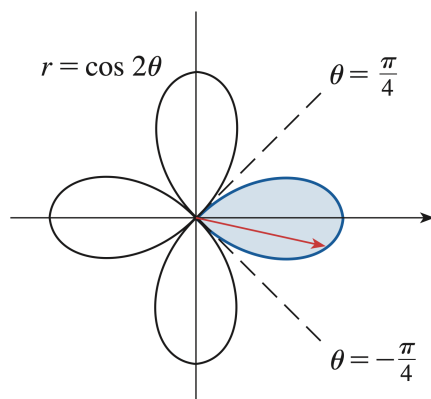


- Divide the interval $[a, b]$ into n subintervals of equal width $\Delta\theta$.
- Let $\theta_0, \theta_1, \dots, \theta_n$ be points in the interval such that $\theta_0 = a$ and $\theta_n = b$.
- Choose sample points θ_i^* in each subinterval.
- What is the approximate area of the sector formed in each small interval $[\theta_{i-1}, \theta_i]$?

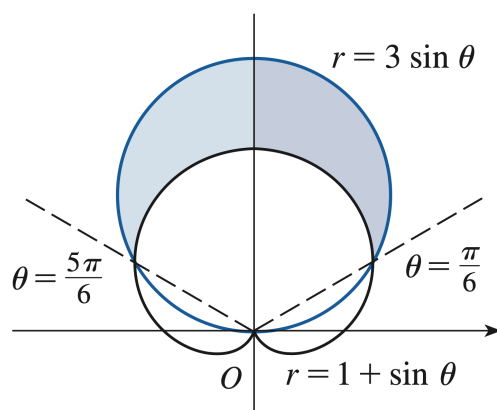
3. Sum over all small sectors to get a formula for the total area.

□

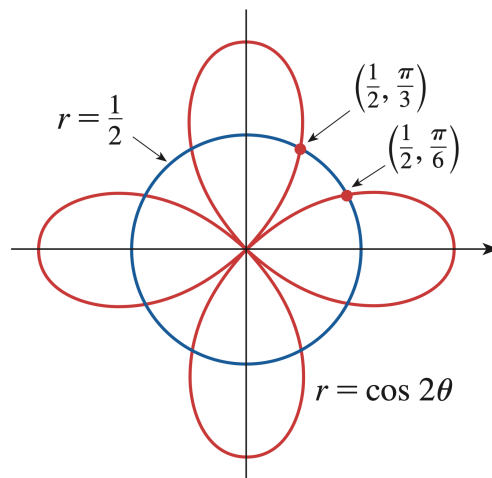
Example. Find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.



Example. Find the area of the region that lies inside the circle $r = 3 \sin \theta$ and outside the cardioid $r = 1 + \sin \theta$.



Example. Find all points of intersection of the curves $r = \cos 2\theta$ and $r = \frac{1}{2}$.



Theorem (Arc Length in Polar Coordinates). Let $r = f(\theta)$ define a smooth polar curve over the interval $a \leq \theta \leq b$. The arc length L of this curve is given by

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

Proof.

1. Express the Cartesian coordinates of the polar curve $r = f(\theta)$ in terms of θ .
2. Differentiate both equations with respect to θ , using the Product Rule.

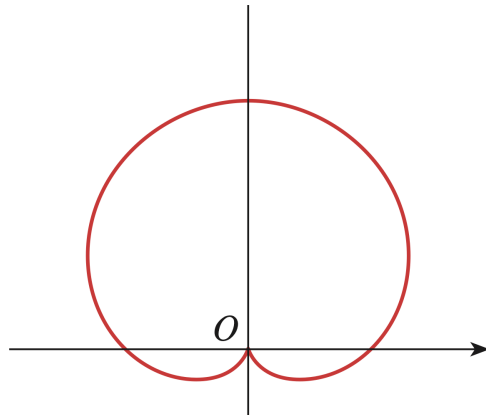
3. The arc length formula for a parametrically defined curve is given by:

$$L = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta.$$

Substitute the expressions for $\frac{dx}{d\theta}$ and $\frac{dy}{d\theta}$ and simplify.

□

Example. Find the length of the cardioid $r = 1 + \sin \theta$.



Theorem (Slope of a Tangent Line to a Polar Curve). Let $r = f(\theta)$ define a polar curve. The slope of the tangent line to the curve is given by

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta} \sin \theta + r \cos \theta}{\frac{dr}{d\theta} \cos \theta - r \sin \theta}.$$

Horizontal tangents occur when $\frac{dy}{d\theta} = 0$ and $\frac{dx}{d\theta} \neq 0$, while vertical tangents occur when $\frac{dx}{d\theta} = 0$ and $\frac{dy}{d\theta} \neq 0$. Furthermore, if the curve passes through the origin ($r = 0$), the slope simplifies to

$$\frac{dy}{dx} = \tan \theta \quad \text{if} \quad \frac{dr}{d\theta} \neq 0.$$

Proof.

1. Treat θ as a parameter and express the Cartesian coordinates in terms of θ .
2. Differentiate both expressions with respect to θ , using the Product Rule.
3. By the chain rule, the slope of the tangent line is given by:
4. At the origin, $r = 0$. In this case, the formula simplifies to:

□

Example.

- (a) For the cardioid $r = 1 + \sin \theta$, find the slope of the tangent line when $\theta = \frac{\pi}{3}$.
- (b) Find the points on the cardioid where the tangent line is horizontal or vertical.

