10.4 Calculus in Polar Coordinates

Area in Polar Coordinates

Theorem (Area of a Polar Region)**.** Let $r = f(\theta)$ be a non-negative, continuous function on the interval $\theta \in [a, b]$, with $b - a \leq 2\pi$. Then the area A of the region bounded by the curve $r = f(\theta)$ and the rays $\theta = a$ and $\theta = b$ is given by

$$A = \frac{1}{2} \int_a^b [f(\theta)]^2 \, d\theta.$$

Theorem. The area of a sector of a circle with radius r and central angle θ (in radians) is given by

$$\text{Area} = \frac{1}{2} r^2 \theta.$$

This result underlies the derivation of the polar area formula.

Tangents and Slopes in Polar Coordinates

Theorem (Slope of the Tangent Line for a Polar Curve)**.** For a polar curve defined by $r = f(\theta)$, the Cartesian coordinates are

$$x = f(\theta) \cos \theta, \quad y = f(\theta) \sin \theta.$$

Differentiating with respect to θ gives

$$\frac{dx}{d\theta} = f'(\theta) \cos \theta - f(\theta) \sin \theta, \quad \frac{dy}{d\theta} = f'(\theta) \sin \theta + f(\theta) \cos \theta.$$

Thus, the slope of the tangent line is

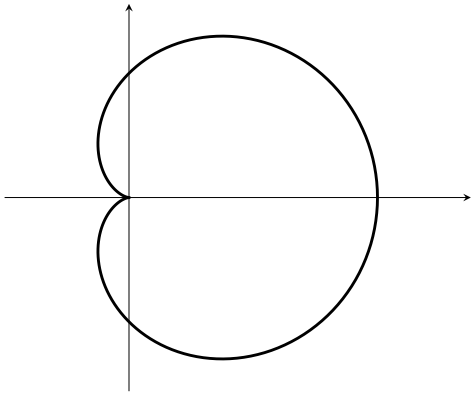
$$\frac{dy}{dx} = \frac{f'(\theta) \sin \theta + f(\theta) \cos \theta}{f'(\theta) \cos \theta - f(\theta) \sin \theta}.$$

Polar Coordinates Problems

Areas Between Polar Curves

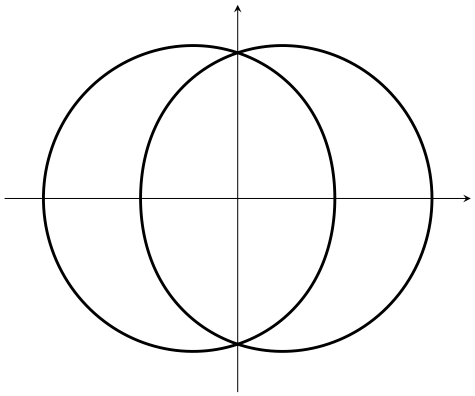
1. Set up an integral to find the area enclosed by the cardioid:

$$r = 2(1 + \cos\theta).$$



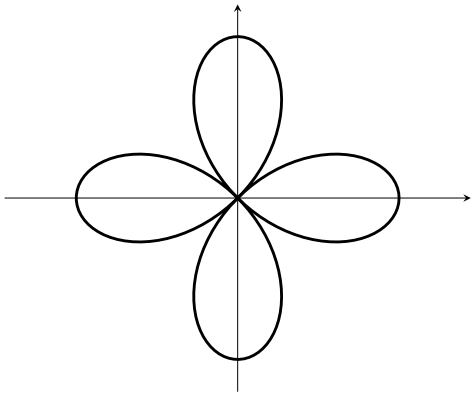
2. Find the area common to both polar curves:

$$r = 3 + \cos\theta, \quad r = 3 - \cos\theta.$$



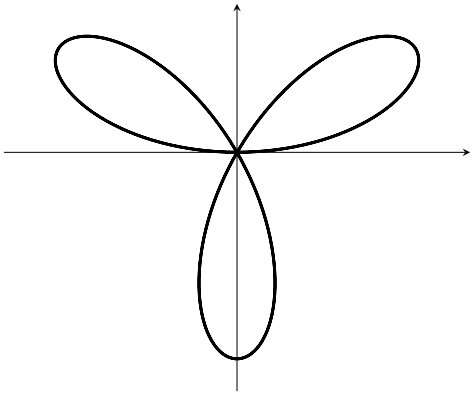
3. Find the area enclosed by the four-leaved rose:

$$r = 3\cos(2\theta).$$

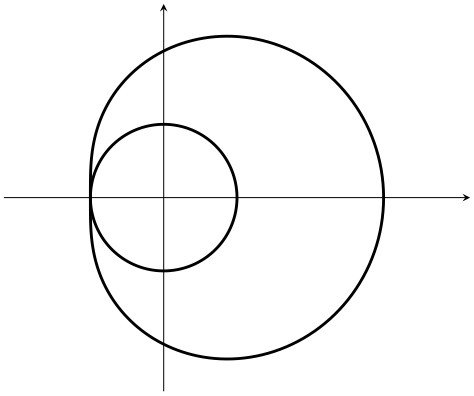


4. Compute the area inside one petal of the rose curve:

$$r = 2\sin(3\theta).$$

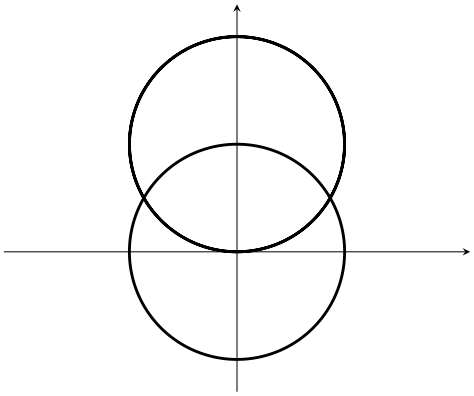


5. Find the area inside $r = 2+\cos\theta$ and outside $r = 1$.



6. Find the area inside:

$$r = 6\sin\theta, \quad r = 3.$$



Tangent Lines and Intersection Points

1. Find all points of intersection of the curves:

$$r = 1 + \sin\theta, \quad r = 1 - \cos\theta.$$

2. Find all points of intersection of the curves:

$$r = 2\cos 2\theta, \quad r = 1.$$

3. Find the slope of the tangent line to the curve:

$$r = 1 + 2\sin\theta, \quad \theta = \frac{\pi}{6}.$$

4. Find points where the tangent line is horizontal or vertical for:

$$r = 2(1 - \cos\theta).$$