10.4 Calculus in Polar Coordinates (Solutions)

Areas Between Polar Curves

1. **Set up an integral to find the area enclosed by the cardioid:**

$$r = 2(1 + \cos\theta).$$

Solution:

- The formula for the area enclosed by a polar curve is:

$$A = \frac{1}{2}\int_a^b r^2\, d\theta.$$

- Substituting $r = 2(1 + \cos\theta)$:

$$\begin{aligned} A &= \frac{1}{2}\int_0^{2\pi} [2(1+\cos\theta)]^2\, d\theta \\ &= \frac{1}{2}\int_0^{2\pi} \left[4(1 + 2\cos\theta + \cos^2\theta)\right] d\theta. \end{aligned}$$

- Expanding the integral:

$$\begin{aligned} A &= \frac{1}{2}\int_0^{2\pi} 4 + 8\cos\theta + 4\cos^2\theta\, d\theta \\ &= \frac{1}{2}\left[\int_0^{2\pi} 4\, d\theta + \int_0^{2\pi} 8\cos\theta\, d\theta + \int_0^{2\pi} 4\cos^2\theta\, d\theta\right]. \end{aligned}$$

- Evaluating each integral:
  - $\int_0^{2\pi} 4\, d\theta = 4(2\pi) = 8\pi.$
  - $\int_0^{2\pi} 8\cos\theta\, d\theta = 0.$
  - Using the identity $\cos^2\theta = \frac{1+\cos 2\theta}{2}$:

$$\int_0^{2\pi} 4\cos^2\theta\, d\theta = 4\int_0^{2\pi} \frac{1+\cos 2\theta}{2} d\theta.$$

  - Splitting the integral:

$$\begin{aligned} \int_0^{2\pi} 4\cos^2\theta\, d\theta &= 4\left[\frac{1}{2}\int_0^{2\pi} 1\, d\theta + \frac{1}{2}\int_0^{2\pi} \cos 2\theta\, d\theta\right] \\ &= 4\left[\frac{1}{2}(2\pi) + \frac{1}{2}(0)\right] = 4[\pi] = 4\pi. \end{aligned}$$

- Summing the results:

$$A = \frac{1}{2}[8\pi + 0 + 4\pi] = \frac{1}{2}(12\pi) = 6\pi.$$

- **Final Answer:**

$$\boxed{6\pi}$$

2. **Find the area common to both polar curves:**

$$r = 3 + \cos\theta, \quad r = 3 - \cos\theta.$$

Solution:

- To find the intersection points, set $r_1 = r_2$:

$$3 + \cos\theta = 3 - \cos\theta.$$

- Solving for θ:

$$\cos\theta = -\cos\theta \quad \Rightarrow \quad \cos\theta = 0.$$

- This occurs at:

$$\theta = \frac{\pi}{2}, \quad \theta = \frac{3\pi}{2}.$$

3. **Find the area enclosed by the four-leaved rose:**

$$r = 3\cos(2\theta).$$

Solution:

- **Determine Limits for One Petal:**

$$3\cos(2\theta) = 0 \quad \Rightarrow \quad \cos(2\theta) = 0.$$

  This yields

$$2\theta = \pm\frac{\pi}{2} \quad \Rightarrow \quad \theta = \pm\frac{\pi}{4},$$

  so one petal is traced when

$$\theta \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right].$$

- **Area of One Petal:**

$$\begin{aligned} A_{\text{petal}} &= \frac{1}{2}\int_{-\pi/4}^{\pi/4} \Big(3\cos(2\theta)\Big)^2 \, d\theta \\ &= \frac{9}{2}\int_{-\pi/4}^{\pi/4} \cos^2(2\theta)\, d\theta. \end{aligned}$$

- **Substitute $u = 2\theta$:**

$$d\theta = \frac{du}{2}, \quad \theta = -\frac{\pi}{4} \Rightarrow u = -\frac{\pi}{2}, \quad \theta = \frac{\pi}{4} \Rightarrow u = \frac{\pi}{2}.$$

  Hence,

$$\begin{aligned} A_{\text{petal}} &= \frac{9}{2}\int_{-\pi/4}^{\pi/4} \cos^2(2\theta)\, d\theta \\ &= \frac{9}{2}\int_{-\pi/2}^{\pi/2} \cos^2(u)\, \frac{du}{2} \\ &= \frac{9}{4}\int_{-\pi/2}^{\pi/2} \cos^2(u)\, du \\ &= \frac{9}{4}\int_{-\pi/2}^{\pi/2} \frac{1+\cos(2u)}{2}\, du \\ &= \frac{9}{8}\int_{-\pi/2}^{\pi/2} \left(1+\cos(2u)\right)\, du \\ &= \frac{9}{8}\left[\left(u + \frac{\sin(2u)}{2}\right)\right]_{-\pi/2}^{\pi/2} \\ &= \frac{9}{8}\cdot\pi \end{aligned}$$

- **Total Area:**

$$A_{\text{total}} = 4 \cdot A_{\text{petal}} = 4 \cdot \frac{9\pi}{8} = \frac{9\pi}{2}.$$

4. **Compute the area inside one petal of the rose curve:**

$$r = 2\sin(3\theta).$$

Solution:

- The formula for the area enclosed by a polar curve is:

$$A = \frac{1}{2}\int_a^b r^2\, d\theta.$$

- A three-petal rose $r = 2\sin(3\theta)$ has three identical petals.
- One petal is traced when θ runs from 0 to $\frac{\pi}{3}$.
- The area of one petal is:

$$A = \frac{1}{2}\int_0^{\pi/3} (2\sin(3\theta))^2\, d\theta.$$

- Expanding the square:

$$A = \frac{1}{2}\int_0^{\pi/3} 4\sin^2(3\theta)\, d\theta.$$

- Using the identity $\sin^2 x = \frac{1-\cos 2x}{2}$:

$$A = \frac{1}{2}\int_0^{\pi/3} 4 \times \frac{1-\cos 6\theta}{2}\, d\theta.$$

- Splitting the integral:

$$A = \frac{1}{2}\left[\int_0^{\pi/3} 2\, d\theta - \int_0^{\pi/3} 2\cos 6\theta\, d\theta\right].$$

- Evaluating:
  - $\int_0^{\pi/3} 2\, d\theta = 2 \times \frac{\pi}{3} = \frac{2\pi}{3}$.
  - $\int_0^{\pi/3} 2\cos 6\theta\, d\theta = 2 \times \frac{\sin 6\theta}{6}\Big|_0^{\pi/3} = 0$.
- So the final area is:

$$A = \frac{1}{2} \times \frac{2\pi}{3} = \frac{\pi}{3}.$$

- **Final Answer:**

$$\boxed{\frac{\pi}{3}}$$

5. **Find the area inside $r = 2 + \cos\theta$ and outside $r = 1$.**

Solution:

The area enclosed between two polar curves is given by:

$$A = \frac{1}{2}\int_a^b \left(r_{\text{outer}}^2 - r_{\text{inner}}^2\right) d\theta.$$

Here, $r_{\text{outer}} = 2 + \cos\theta$ and $r_{\text{inner}} = 1$.

Step 1: Find the Limits of Integration

The curves intersect when:

$$2 + \cos\theta = 1.$$

Solving for θ: $\cos\theta = -1 \quad \Rightarrow \quad \theta = \pi$. Since we are computing the total enclosed area, we integrate from $\theta = 0$ to $\theta = 2\pi$.

Step 2: Integrate

$$\begin{aligned} A &= \frac{1}{2}\int_0^{2\pi} \left((2+\cos\theta)^2 - 1^2\right) d\theta \\ &= \frac{1}{2}\int_0^{2\pi} \left(4 + 4\cos\theta + \cos^2\theta - 1\right) d\theta \\ &= \frac{1}{2}\int_0^{2\pi} \left(3 + 4\cos\theta + \cos^2\theta\right) d\theta \\ &= \frac{1}{2}\left[\int_0^{2\pi} 3\, d\theta + \int_0^{2\pi} 4\cos\theta\, d\theta + \int_0^{2\pi} \cos^2\theta\, d\theta\right]. \end{aligned}$$

Computing each integral:

$$\int_0^{2\pi} 3\, d\theta = 3(2\pi) = 6\pi,$$
$$\int_0^{2\pi} 4\cos\theta\, d\theta = 0.$$

Using $\cos^2\theta = \frac{1+\cos 2\theta}{2}$, we rewrite:

$$\int_0^{2\pi} \cos^2\theta\, d\theta = \int_0^{2\pi} \frac{1+\cos 2\theta}{2} d\theta.$$

Splitting:

$$\int_0^{2\pi} \frac{1+\cos 2\theta}{2} d\theta = \frac{1}{2}\int_0^{2\pi} 1\, d\theta + \frac{1}{2}\int_0^{2\pi} \cos 2\theta\, d\theta.$$

Since $\int_0^{2\pi} \cos 2\theta\, d\theta = 0$, we obtain:

$$\frac{1}{2} \cdot 2\pi = \pi.$$

Step 3: Compute the Final Result

$$\begin{aligned} A &= \frac{1}{2}(6\pi + 0 + \pi) \\ &= \frac{7\pi}{2}. \end{aligned}$$

6. **Find the area inside $r = 6\sin\theta$ and outside $r = 3$.**

Solution:

The area enclosed between two polar curves is given by:

$$A = \frac{1}{2}\int_a^b \left(r_{\text{outer}}^2 - r_{\text{inner}}^2\right) d\theta.$$

Here, $r_{\text{outer}} = 6\sin\theta$ and $r_{\text{inner}} = 3$.

Step 1: Find the Limits of Integration

The curves intersect when:

$$6\sin\theta = 3 \quad \Rightarrow \quad \sin\theta = \frac{1}{2}, \quad \theta = \frac{\pi}{6}, \quad \frac{5\pi}{6}.$$

We integrate from $\theta = \frac{\pi}{6}$ to $\theta = \frac{5\pi}{6}$.

Step 2: Set Up the Integral

$$\begin{aligned} A &= \frac{1}{2}\int_{\pi/6}^{5\pi/6} \left((6\sin\theta)^2 - 3^2\right) d\theta \\ &= \frac{1}{2}\int_{\pi/6}^{5\pi/6} \left(36\sin^2\theta - 9\right) d\theta. \end{aligned}$$

Using $\sin^2\theta = \frac{1-\cos 2\theta}{2}$, we substitute:

$$\begin{aligned} A &= \frac{1}{2}\int_{\pi/6}^{5\pi/6} \left(36 \cdot \frac{1-\cos 2\theta}{2} - 9\right) d\theta \\ &= \frac{1}{2}\int_{\pi/6}^{5\pi/6} (9 - 18\cos 2\theta)\, d\theta. \end{aligned}$$

Step 3: Evaluate the Integral

Computing each term:

$$\int_{\pi/6}^{5\pi/6} 9\, d\theta = 9\left(\frac{5\pi}{6} - \frac{\pi}{6}\right) = 9 \cdot \frac{4\pi}{6} = 6\pi,$$

$$\begin{aligned} \int_{\pi/6}^{5\pi/6} 18\cos 2\theta\, d\theta &= 18 \cdot \frac{\sin 2\theta}{2}\Big|_{\pi/6}^{5\pi/6} \\ &= 9\left(\sin\frac{5\pi}{3} - \sin\frac{\pi}{3}\right) \\ &= -9\sqrt{3} \end{aligned}$$

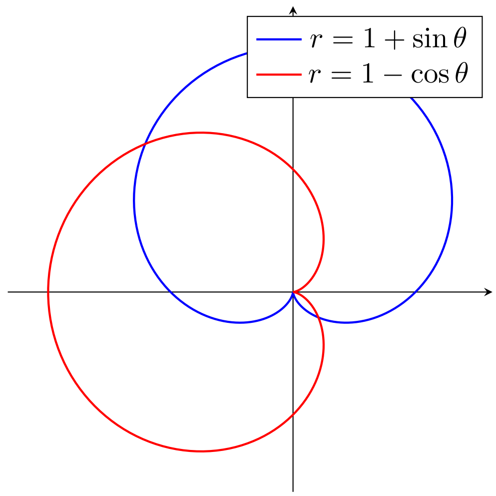
Step 4: Compute the Final Result

$$\begin{aligned} A &= \frac{1}{2}\left(6\pi + 9\sqrt{3}\right) \\ &= 3\pi + \frac{9\sqrt{3}}{2}. \end{aligned}$$

Tangent Lines and Arc Length

1. Find all points of intersection of the curves:

$$r = 1 + \sin\theta, \quad r = 1 - \cos\theta.$$



Solution:

Step 1: Justifying the Range $[0, 2\pi]$

While the equation $r = 1 + \sin\theta$ completes one full trace over $0 \le \theta \le \pi$, and $r = 1 - \cos\theta$ also completes its trace over $[0, \pi]$, we must consider θ in the full range $[0, 2\pi]$.

The reason is that in polar coordinates, a curve may intersect itself or intersect another curve at different angles, sometimes occurring at θ values outside $[0, \pi]$. Additionally, the same point can have multiple representations in polar form, meaning that intersections might not be obvious just by looking at $[0, \pi]$.

Step 2: Finding Intersection Points

To find intersections, we set the equations equal to each other:

$$1 + \sin\theta = 1 - \cos\theta.$$

Simplifying:

$$\sin\theta + \cos\theta = 0.$$

Dividing by $\cos\theta$ (where valid):

$$\tan\theta = -1.$$

The general solutions to $\tan\theta = -1$ are:

$$\theta = \frac{3\pi}{4} + k\pi, \quad k \in \mathbb{Z}.$$

Restricting to $[0, 2\pi]$, the valid solutions are:

$$\theta = \frac{3\pi}{4}, \quad \theta = \frac{7\pi}{4}.$$

Step 3: Compute r at These Intersections

Substituting into $r = 1 + \sin\theta$:

$$r = 1 + \sin\frac{3\pi}{4} = 1 + \frac{\sqrt{2}}{2} = \frac{2 + \sqrt{2}}{2}.$$

$$r = 1 + \sin\frac{7\pi}{4} = 1 - \frac{\sqrt{2}}{2} = \frac{2 - \sqrt{2}}{2}.$$

Thus, two intersection points are:

$$\left(\frac{2 + \sqrt{2}}{2}, \frac{3\pi}{4}\right), \quad \left(\frac{2 - \sqrt{2}}{2}, \frac{7\pi}{4}\right).$$

Step 4: Checking for the Origin as an Intersection

A point is at the origin if $r = 0$ for some θ. Setting $r = 0$ in both equations:

- From $r = 1 + \sin\theta$, setting $1 + \sin\theta = 0$ gives:

$$\sin\theta = -1 \Rightarrow \theta = \frac{3\pi}{2}.$$

- From $r = 1 - \cos\theta$, setting $1 - \cos\theta = 0$ gives:

$$\cos\theta = 1 \Rightarrow \theta = 0, 2\pi.$$

Since both equations independently achieve $r = 0$ at different values of θ, they both pass through the origin, meaning $(0, 0)$ must be included as an intersection.

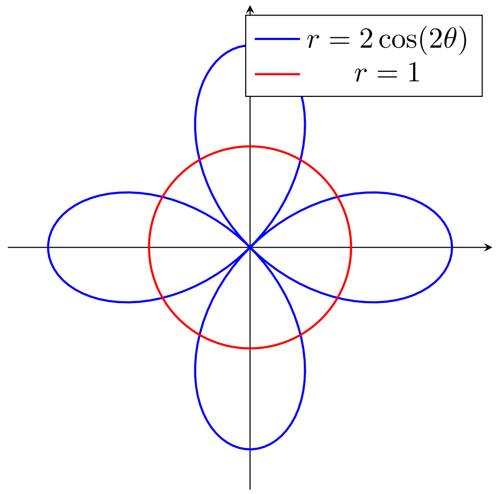
Step 5: Final Answer Including the Origin

The full set of intersection points is:

$$\left(\frac{2 + \sqrt{2}}{2}, \frac{3\pi}{4}\right), \quad \left(\frac{2 - \sqrt{2}}{2}, \frac{7\pi}{4}\right), \quad (0, 0).$$

2. Find all points of intersection of the curves:

$$r = 2\cos 2\theta, \quad r = 1.$$



Solution:

Step 1: Solve for Intersections with $r = 1$

To find where the curves intersect, we first solve:

$$2\cos 2\theta = 1.$$

Solving for θ:

$$\cos 2\theta = \frac{1}{2}.$$

The general solutions for $\cos x = \frac{1}{2}$ are:

$$2\theta = \pm\frac{\pi}{3} + 2k\pi, \quad k \in \mathbb{Z}.$$

Dividing by 2:

$$\theta = \pm\frac{\pi}{6} + k\pi.$$

Restricting to $0 \le \theta < 2\pi$, the valid solutions are:

$$\theta = \frac{\pi}{6}, \quad \theta = \frac{5\pi}{6}, \quad \theta = \frac{7\pi}{6}, \quad \theta = \frac{11\pi}{6}.$$

Since we set $r = 1$, the intersection points are:

$$(1, \frac{\pi}{6}), \quad (1, \frac{5\pi}{6}), \quad (1, \frac{7\pi}{6}), \quad (1, \frac{11\pi}{6}).$$

Step 2: Solve for Intersections with $r = -1$

A point (r, θ) is equivalent to $(-r, \theta + \pi)$, so we solve:

$$2\cos 2\theta = -1.$$

Dividing by 2:

$$\cos 2\theta = -\frac{1}{2}.$$

The general solutions for $\cos x = -\frac{1}{2}$ are:

$$2\theta = \pm\frac{2\pi}{3} + 2k\pi, \quad k \in \mathbb{Z}.$$

Dividing by 2:

$$\theta = \pm\frac{\pi}{3} + k\pi.$$

Restricting to $0 \leq \theta < 2\pi$, the valid solutions are:

$$\theta = \frac{\pi}{3}, \quad \theta = \frac{2\pi}{3}, \quad \theta = \frac{4\pi}{3}, \quad \theta = \frac{5\pi}{3}.$$

Since these correspond to $r = -1$, we convert to equivalent positive r representations:

$$(-1, \frac{\pi}{3}) = (1, \frac{4\pi}{3}),$$

$$(-1, \frac{2\pi}{3}) = (1, \frac{5\pi}{3}),$$

$$(-1, \frac{4\pi}{3}) = (1, \frac{\pi}{3}),$$

$$(-1, \frac{5\pi}{3}) = (1, \frac{2\pi}{3}).$$

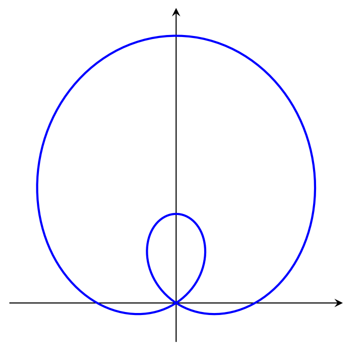
Final Answer: The full set of intersection points is:

$$(1, \frac{\pi}{6}), \quad (1, \frac{5\pi}{6}), \quad (1, \frac{7\pi}{6}), \quad (1, \frac{11\pi}{6}),$$

$$(1, \frac{\pi}{3}), \quad (1, \frac{2\pi}{3}), \quad (1, \frac{4\pi}{3}), \quad (1, \frac{5\pi}{3}).$$

3. Find the slope of the tangent line to the curve:

$$r = 1 + 2\sin\theta, \quad \theta = \frac{\pi}{6}.$$



Solution:

The slope of the tangent line to a polar curve is given by:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}.$$

Step 1: Compute Cartesian Coordinates The Cartesian coordinates of a polar curve are:

$$x = r\cos\theta, \quad y = r\sin\theta.$$

Substituting $r = 1 + 2\sin\theta$:

$$x = (1 + 2\sin\theta)\cos\theta, \quad y = (1 + 2\sin\theta)\sin\theta.$$

Step 2: Compute Derivatives Differentiating x with respect to θ:

$$\frac{dx}{d\theta} = (2\cos\theta)\cos\theta - (1 + 2\sin\theta)\sin\theta.$$

Differentiating y with respect to θ:

$$\frac{dy}{d\theta} = (2\cos\theta)\sin\theta + (1 + 2\sin\theta)\cos\theta.$$

Step 3: Evaluate at $\theta = \frac{\pi}{6}$ First, compute necessary trigonometric values:

$$\sin\frac{\pi}{6} = \frac{1}{2}, \quad \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}.$$

Compute r at $\theta = \frac{\pi}{6}$:

$$r = 1 + 2\sin\frac{\pi}{6} = 1 + 2\cdot\frac{1}{2} = 2.$$

Compute $\frac{dx}{d\theta}$:

$$\begin{aligned}
\frac{dx}{d\theta} &= (2\cos\frac{\pi}{6})\cos\frac{\pi}{6} - (1 + 2\sin\frac{\pi}{6})\sin\frac{\pi}{6} \\
&= \left(2\cdot\frac{\sqrt{3}}{2}\right)\frac{\sqrt{3}}{2} - (2)\frac{1}{2} \\
&= \left(\sqrt{3}\right)\frac{\sqrt{3}}{2} - 1 \\
&= \frac{3}{2} - 1 = \frac{1}{2}.
\end{aligned}$$

Compute $\frac{dy}{d\theta}$:

$$\begin{aligned}
\frac{dy}{d\theta} &= (2\cos\frac{\pi}{6})\sin\frac{\pi}{6} + (1 + 2\sin\frac{\pi}{6})\cos\frac{\pi}{6} \\
&= \left(2\cdot\frac{\sqrt{3}}{2}\right)\frac{1}{2} + (2)\frac{\sqrt{3}}{2} \\
&= \left(\sqrt{3}\right)\frac{1}{2} + \sqrt{3} \\
&= \frac{\sqrt{3}}{2} + \sqrt{3} = \frac{3\sqrt{3}}{2}.
\end{aligned}$$

Step 4: Compute the Slope

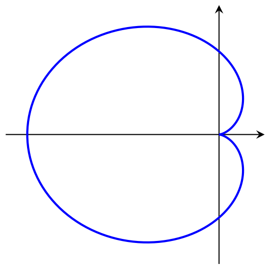
$$\frac{dy}{dx} = \frac{\frac{3\sqrt{3}}{2}}{\frac{1}{2}} = 3\sqrt{3}.$$

Final Answer:

$$\frac{dy}{dx} = 3\sqrt{3}.$$

4. Find points where the tangent line is horizontal or vertical for:

$$r = 2(1 - \cos\theta).$$



Solution:

The slope of the tangent line to a polar curve is given by:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}.$$

Horizontal tangents occur where $\frac{dy}{d\theta} = 0$ and $\frac{dx}{d\theta} \neq 0$. Vertical tangents occur where $\frac{dx}{d\theta} = 0$ and $\frac{dy}{d\theta} \neq 0$.

Step 1: Compute First Derivatives

The Cartesian coordinates of the curve are:

$$x = r\cos\theta = 2(1-\cos\theta)\cos\theta,$$
$$y = r\sin\theta = 2(1-\cos\theta)\sin\theta.$$

Differentiating x with respect to θ:

$$\begin{aligned}\frac{dx}{d\theta} &= 2\left((1-\cos\theta)(-\sin\theta) + \cos\theta\sin\theta\right)\\ &= 2\left(-\sin\theta + 2\cos\theta\sin\theta\right)\\ &= 2\sin\theta(2\cos\theta - 1).\end{aligned}$$

Differentiating y with respect to θ:

$$\begin{aligned}\frac{dy}{d\theta} &= 2\left((1-\cos\theta)\cos\theta + \sin\theta\sin\theta\right)\\ &= 2\left(\cos\theta - \cos^2\theta + \sin^2\theta\right)\\ &= 2\left(\cos\theta - \cos 2\theta\right).\end{aligned}$$

Step 2: Solve $\frac{dy}{d\theta} = 0$

Setting $\frac{dy}{d\theta} = 0$:

$$\cos\theta - \cos 2\theta = 0.$$

Rearranging:

$$\cos\theta = \cos 2\theta.$$

The general solutions to $\cos A = \cos B$ are:

$$A = B + 2k\pi \quad \text{or} \quad A = -B + 2k\pi.$$

Applying $A = \theta$ and $B = 2\theta$, we get:

$$\theta = \frac{2k\pi}{3}.$$

Restricting to $0 \leq \theta \leq 2\pi$, the valid solutions are:

$$\theta = 0, \quad \frac{2\pi}{3}, \quad \frac{4\pi}{3}, \quad 2\pi.$$

Step 3: Solve $\frac{dx}{d\theta} = 0$

Setting $\frac{dx}{d\theta} = 0$:

$$2\sin\theta(2\cos\theta - 1) = 0.$$

Solving each factor separately:

- $\sin\theta = 0$ gives:

$$\theta = 0, \pi, 2\pi.$$

- $2\cos\theta - 1 = 0$ gives:

$$\cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}, \quad \frac{5\pi}{3}.$$

Step 4: Apply L'Hôpital's Rule at $\theta = 0$ and $\theta = 2\pi$

At $\theta = 0$ and $\theta = 2\pi$, both $\frac{dx}{d\theta} = 0$ and $\frac{dy}{d\theta} = 0$, so we differentiate again:

$$\frac{d^2x}{d\theta^2} = 2(2\cos^2\theta - \cos\theta - 1 + \cos 2\theta).$$

$$\frac{d^2y}{d\theta^2} = 2(-\sin\theta + 2\sin 2\theta).$$

Evaluating at $\theta = 0$:

$$\frac{d^2x}{d\theta^2} = 2, \quad \frac{d^2y}{d\theta^2} = 0.$$

Since $\frac{d^2x}{d\theta^2} \neq 0$ and $\frac{d^2y}{d\theta^2} = 0$, $\theta = 0$ gives a horizontal tangent.

Evaluating at $\theta = 2\pi$:

$$\frac{d^2x}{d\theta^2} = 2, \quad \frac{d^2y}{d\theta^2} = 0.$$

Since $\frac{d^2x}{d\theta^2} \neq 0$ and $\frac{d^2y}{d\theta^2} = 0$, $\theta = 2\pi$ gives a horizontal tangent.

Final Answer:

The curve has horizontal tangents at:

$$\theta = 0, \quad \frac{2\pi}{3}, \quad \frac{4\pi}{3}, \quad 2\pi.$$

The curve has vertical tangents at:

$$\theta = \pi, \quad \frac{\pi}{3}, \quad \frac{5\pi}{3}.$$