10.4 Calculus in Polar Coordinates

Theorem. Let $r = f(\theta)$ be a continuous function defining a curve in polar coordinates. The area A enclosed by the curve between the angles $\theta = a$ and $\theta = b$ is given by

$$A = \int_a^b \frac{1}{2} r^2 \, d\theta.$$

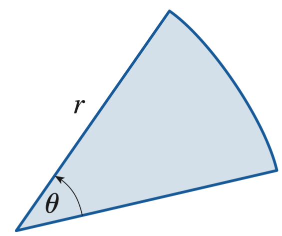
The area is interpreted as being swept out by a rotating ray through the origin O that starts at angle a and ends at angle b.

Proof.

1. What is the area of a sector of a circle?

A full circle of radius r has area πr^2

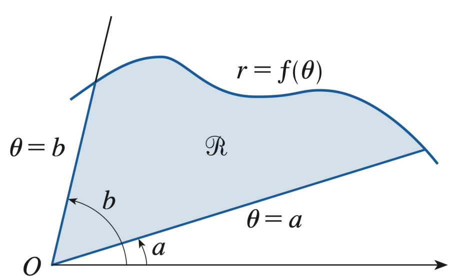
A sector with angle θ (in radians) is $\frac{\theta}{2\pi}$ of the full circle

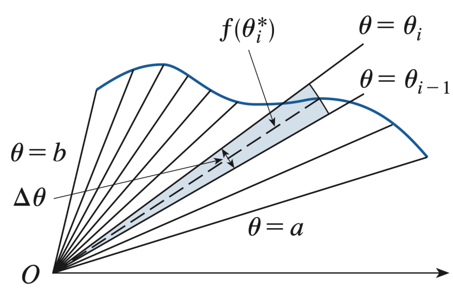


Hence $A = \frac{\theta}{2\pi} \cdot \pi r^2 = \frac{1}{2} r^2 \theta$

$f(\theta) \geq 0$ and $0 < b - a \leq 2\pi$

2. Consider the area enclosed by the polar curve $r = f(\theta)$ between angles $\theta = a$ and $\theta = b$.





- Divide the interval $[a, b]$ into n subintervals of equal width $\Delta\theta$.
- Let $\theta_0, \theta_1, \ldots, \theta_n$ be points in the interval such that $\theta_0 = a$ and $\theta_n = b$.
- Choose sample points θ_i^* in each subinterval.
- What is the approximate area of the sector formed in each small interval $[\theta_{i-1}, \theta_i]$?

Each small region is approximately a circular sector with radius $f(\theta_i^*)$

$A_i \approx \frac{1}{2} [f(\theta_i^*)]^2 \Delta\theta$

3. Sum over all small sectors to get a formula for the total area.

$$A \approx \sum_{i=1}^{n} \frac{1}{2}\left[f(\theta_i^*)\right]^2 \Delta\theta$$

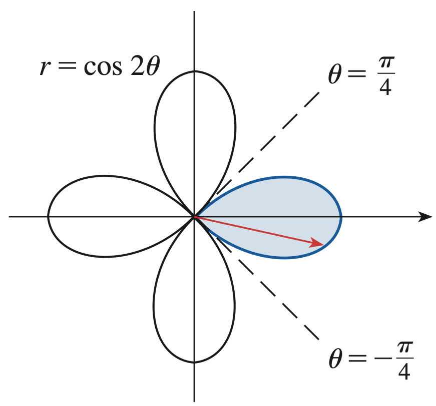
def. of integral

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2}\left[f(\theta_i^*)\right]^2 \Delta\theta = \int_a^b \frac{1}{2}\left[f(\theta)\right]^2 d\theta$$

Since $r = f(\theta)$, we often write $A = \int_a^b \frac{1}{2} r^2 \, d\theta$

□

Example. Find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.



The region enclosed by one loop is swept out as θ moves from $\frac{-\pi}{4}$ to $\frac{\pi}{4}$

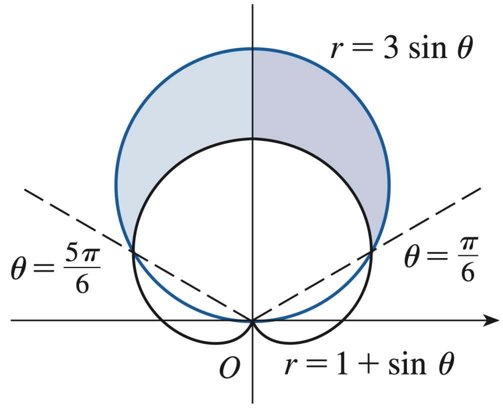
$\cos^2(\theta) = \frac{1}{2}(1 + \cos 2\theta)$

$$A = \int_a^b \frac{1}{2} r^2 \, d\theta = \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2(2\theta)\, d\theta = \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2}\left(1 + \cos(4\theta)\right) d\theta$$

$$= \frac{1}{4}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 1 \, d\theta + \frac{1}{4}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos(4\theta)\, d\theta = \frac{1}{4}\left[\theta\right]_{-\pi/4}^{\pi/4} + \cancelto{0}{\frac{1}{4}\left[\frac{1}{4}\sin(4\theta)\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}}$$

$$= \frac{1}{4}\left(\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right) = \frac{\pi}{8}$$

Example. Find the area of the region that lies inside the circle $r = 3\sin\theta$ and outside the cardioid $r = 1 + \sin\theta$.



Note: we missed the origin.

- On $3\sin\theta$, it occurs at $(0,0)$ and $(0,\pi)$
- On $1+\sin\theta$, it occurs at $(0, \frac{3\pi}{2})$

Draw the graphs to make sure you get every intersection point

The curves intersect when $3\sin\theta = 1 + \sin\theta$

$\Rightarrow 2\sin\theta = 1$

$\Rightarrow \sin\theta = \frac{1}{2}$, so $\theta = \frac{\pi}{6}$ or $\frac{5\pi}{6}$

$$A = \frac{1}{2}\int_{\pi/6}^{5\pi/6} (3\sin\theta)^2 \, d\theta - \frac{1}{2}\int_{\pi/6}^{5\pi/6} (1+\sin\theta)^2 \, d\theta$$

$$= \frac{1}{2}\int_{\pi/6}^{5\pi/6} 8\sin^2\theta - 2\sin\theta - 1 \, d\theta$$

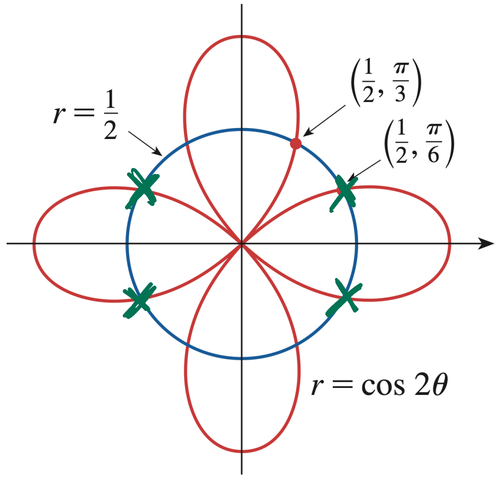
> Trig identity

$$= \frac{1}{2}\int_{\pi/6}^{5\pi/6} 8 \cdot \frac{1}{2}(1 - \cos 2\theta) - 2\sin\theta - 1 \, d\theta$$

$$= \frac{1}{2}\Big[3\theta - 2\sin 2\theta + 2\cos\theta\Big]_{\pi/6}^{5\pi/6}$$

$$= \pi$$

Example. Find all points of intersection of the curves $r = \cos 2\theta$ and $r = \frac{1}{2}$.



Solve $\cos 2\theta = \frac{1}{2}$

$2\theta = \pm \frac{\pi}{3} + 2n\pi$ for integers n

$\theta = \pm \frac{\pi}{6} + n\pi$ for integers n

The values in $[0, 2\pi]$ are $\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

The intersection points are $\left(\frac{1}{2}, \frac{\pi}{6}\right), \left(\frac{1}{2}, \frac{5\pi}{6}\right), \left(\frac{1}{2}, \frac{7\pi}{6}\right), \left(\frac{1}{2}, \frac{11\pi}{6}\right)$

From the graph, there are 4 more. These occur when $r = -\frac{1}{2}$.

↳ In polar, both $r = \frac{1}{2}$ and $r = -\frac{1}{2}$ are circles of radius $\frac{1}{2}$

Intersecting $r = \cos(2\theta)$ with $r = -\frac{1}{2}$ gives the following points:

$\left(\frac{1}{2}, \frac{\pi}{3}\right), \left(\frac{1}{2}, \frac{2\pi}{3}\right), \left(\frac{1}{2}, \frac{4\pi}{3}\right), \left(\frac{1}{2}, \frac{5\pi}{3}\right)$

Theorem (Arc Length in Polar Coordinates)**.** Let $r = f(\theta)$ define a smooth polar curve over the interval $a \le \theta \le b$. The arc length L of this curve is given by

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

Proof.

1. Express the Cartesian coordinates of the polar curve $r = f(\theta)$ in terms of θ.

$x = r\cos\theta = f(\theta)\cos\theta$ $\qquad$ $y = r\sin\theta = f(\theta)\sin\theta$

2. Differentiate both equations with respect to θ, using the Product Rule.

$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta$ $\qquad$ $\frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$

3. The arc length formula for a parametrically defined curve is given by:

$$L = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta.$$

Substitute the expressions for $\frac{dx}{d\theta}$ and $\frac{dy}{d\theta}$ and simplify.

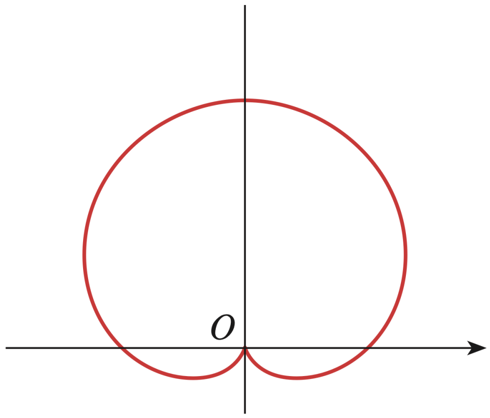
$$L = \int_a^b \sqrt{\underline{\left(\frac{dr}{d\theta}\cos\theta - r\sin\theta\right)^2} + \underline{\left(\frac{dr}{d\theta}\sin\theta + r\cos\theta\right)^2}}\, d\theta$$

$$L = \int_a^b \sqrt{\underline{\left(\frac{dr}{d\theta}\right)^2\cos^2\theta - 2r\frac{dr}{d\theta}\sin\theta\cos\theta + r^2\sin^2\theta} + \underline{\left(\frac{dr}{d\theta}\right)^2\sin^2\theta + 2r\frac{dr}{d\theta}\sin\theta\cos\theta + r^2\cos^2\theta}}\, d\theta$$

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

□

Example. Find the length of the cardioid $r = 1 + \sin\theta$.



$$L = \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$\frac{dr}{d\theta} = \cos\theta$$

$$L = \int_0^{2\pi} \sqrt{(1+\sin\theta)^2 + \cos^2\theta}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{2 + 2\sin\theta}\, d\theta$$

To continue... multiply by $\dfrac{\sqrt{2-2\sin\theta}}{\sqrt{2-2\sin\theta}}$

Final answer: $L = 8$

Theorem (Slope of a Tangent Line to a Polar Curve)**.** Let $r = f(\theta)$ define a polar curve. The slope of the tangent line to the curve is given by

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}.$$

Horizontal tangents occur when $\frac{dy}{d\theta} = 0$ and $\frac{dx}{d\theta} \neq 0$, while vertical tangents occur when $\frac{dx}{d\theta} = 0$ and $\frac{dy}{d\theta} \neq 0$. Furthermore, if the curve passes through the origin ($r = 0$), the slope simplifies to

$$\frac{dy}{dx} = \tan\theta \quad \text{if} \quad \frac{dr}{d\theta} \neq 0.$$

Proof.

1. Treat θ as a parameter and express the Cartesian coordinates in terms of θ.

$$x = r\cos\theta = f(\theta)\cos\theta \qquad y = r\sin\theta = f(\theta)\sin\theta$$

2. Differentiate both expressions with respect to θ, using the Product Rule.

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta \qquad \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

3. By the chain rule, the slope of the tangent line is given by:

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{dr/d\theta\ \sin\theta + r\cos\theta}{dr/d\theta\ \cos\theta - r\sin\theta}$$

4. At the origin, $r = 0$. In this case, the formula simplifies to:

$$\frac{dy}{dx} = \frac{dr/d\theta\ \sin\theta}{dr/d\theta\ \cos\theta} = \tan\theta\text{, provided } dr/d\theta \neq 0$$

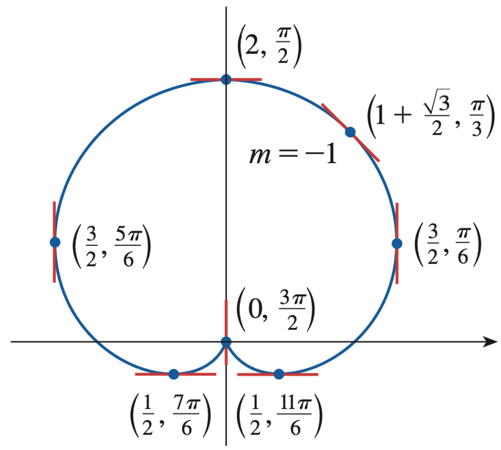
$\square$

Example.

(a) For the cardioid $r = 1 + \sin\theta$, find the slope of the tangent line when $\theta = \frac{\pi}{3}$.

(b) Find the points on the cardioid where the tangent line is horizontal or vertical.

$$\frac{dr}{d\theta} = \cos\theta$$



(a)

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} = \frac{\cos\theta\sin\theta + (1+\sin\theta)\cos\theta}{\cos^2\theta - (1+\sin\theta)\sin\theta} = \frac{\cos\theta\,(1+2\sin\theta)}{(1+\sin\theta)(1-2\sin\theta)}$$

when $\theta = \frac{\pi}{3}$, $\quad \frac{dy}{dx} = \frac{\cos(\pi/3)(1+2\sin\pi/3)}{(1+\sin\pi/3)(1-2\sin\pi/3)} = -1$

(b)

$\frac{dy}{d\theta} = 0$ when $\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$ ← H. tangents

$\frac{dx}{d\theta} = 0$ when $\theta = \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$ ← V. tangents

At $\frac{3\pi}{2}$, use L'Hôpital's rule:

$$\left.\begin{array}{l} \lim_{\theta\to 3\pi/2^-} \frac{dy}{dx} \text{ is } \infty \\ \lim_{\theta\to 3\pi/2^+} \frac{dy}{dx} \text{ is } -\infty \end{array}\right\} \text{V. tangent}$$