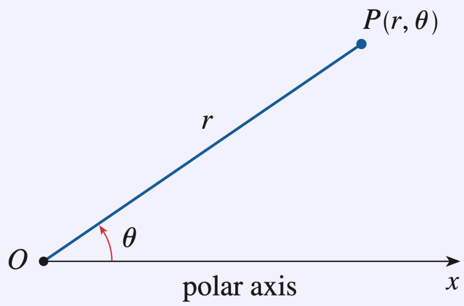
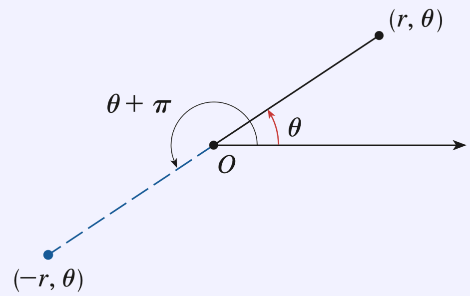
10.3 Polar Coordinates

Definition. The polar coordinate system is defined by:

- A point called the **pole** (or origin), denoted by O.
- A ray starting from O, called the **polar axis**, usually drawn horizontally to the right.



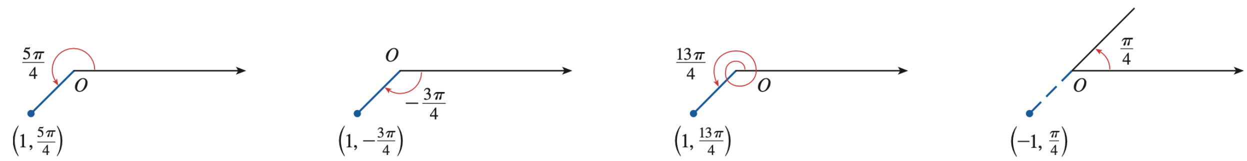


- A point P in the plane is represented by the ordered pair (r, θ), where:
  - r is the distance from O to P.
  - θ is the angle (in radians) between the polar axis and the line OP.
- An angle is positive if measured counterclockwise and negative if measured clockwise.
- If $r < 0$, the point (r, θ) lies on the opposite side of the pole, at the same distance $|r|$.

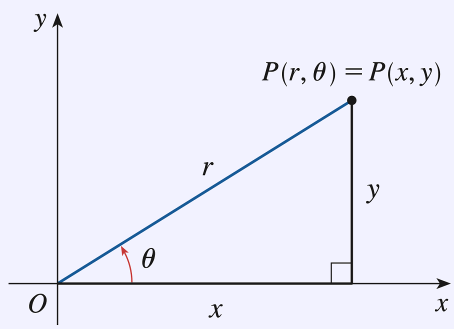
Example. Plot the points with the following polar coordinates:

(a) $(1, \frac{5\pi}{4})$ (b) $(2, 3\pi)$ (c) $(2, -\frac{2\pi}{3})$ (d) $(-3, \frac{3\pi}{4})$

Remark. In the Cartesian coordinate system every point has only one representation, but in the polar coordinate system each point has many representations. For instance, the point $(1, 5\pi/4)$ could be written as $(1, -3\pi/4)$ or $(1, 13\pi/4)$ or $(-1, \pi/4)$.



Theorem. The relationship between polar and Cartesian coordinates is given by:



Example. Convert the point $(2, \frac{\pi}{3})$ from polar to Cartesian coordinates.

Example. Convert the Cartesian point $(-1, -\sqrt{3})$ to polar coordinates.

> **Definition.** The graph of a polar equation $r = f(\theta)$, or more generally $F(r, \theta) = 0$, consists of all points P that have at least one polar representation (r, θ) whose coordinates satisfy the equation.

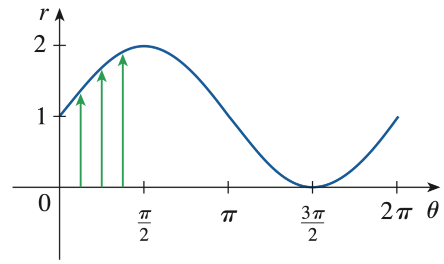
Example. What curve is represented by the polar equation $r = 2$?

Example. Sketch the polar curve $\theta = 1$.

Example. Sketch the curve $r = 2\cos\theta$ and convert it to a Cartesian equation.

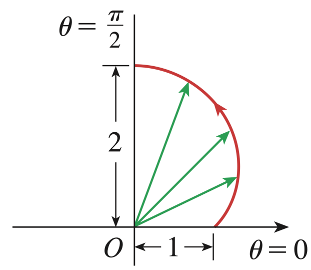
$\boldsymbol{\theta}$	$\boldsymbol{r = 2\cos\theta}$
0	2
$\pi/6$	$\sqrt{3}$
$\pi/4$	$\sqrt{2}$
$\pi/3$	1
$\pi/2$	0
$2\pi/3$	-1
$3\pi/4$	$-\sqrt{2}$
$5\pi/6$	$-\sqrt{3}$
π	-2

Example. Sketch the curve $r = 1 + \sin\theta$.

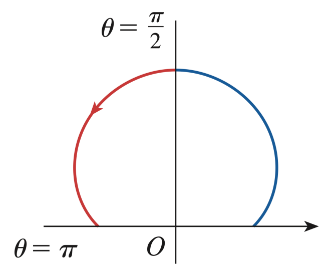


- In Cartesian coordinates, graph the behavior of r for $0 \le \theta \le 2\pi$.

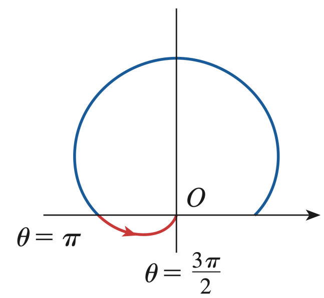
- Analyze the behavior of r for $0 \le \theta \le \pi/2$.



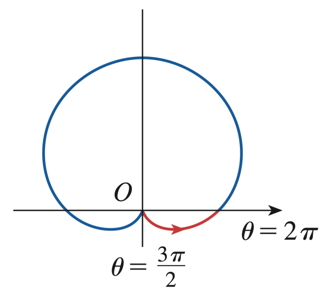
- Analyze the behavior of r for $\pi/2 \le \theta \le \pi$.



- Analyze the behavior of r for $\pi \le \theta \le 3\pi/2$.

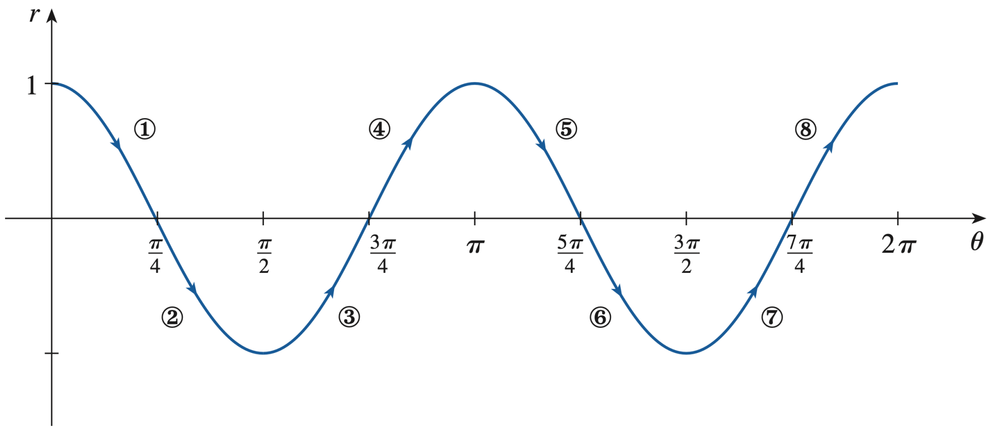


- Analyze the behavior of r for $3\pi/2 \le \theta \le 2\pi$.



Example. Sketch the curve $r = \cos 2\theta$.

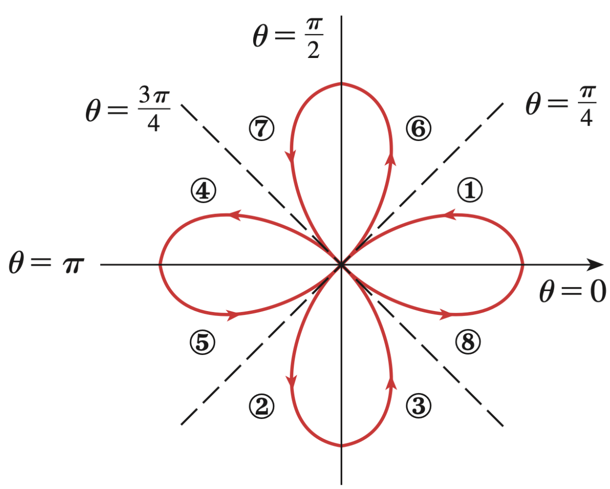
- Sketch $r = \cos 2\theta$ for $\theta \in [0, 2\pi]$. This helps us visualize the values of r as θ varies.



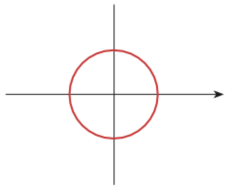
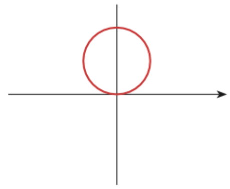
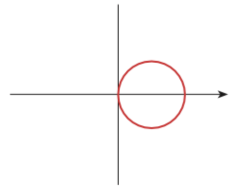
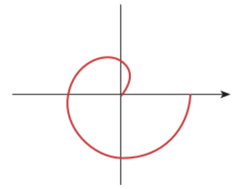
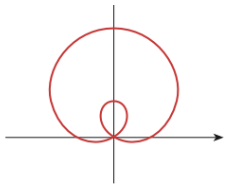
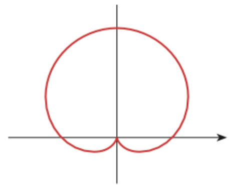
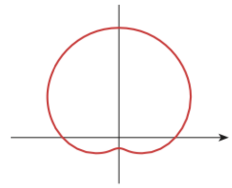
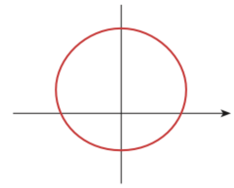
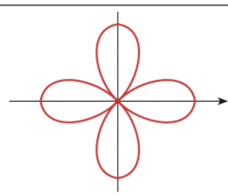
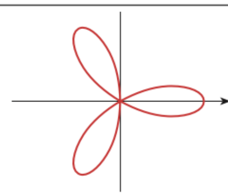
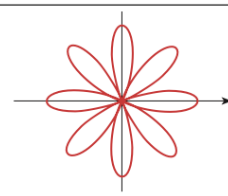
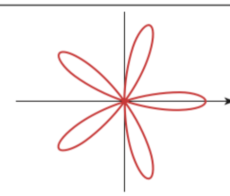
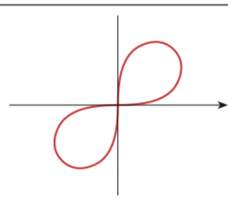
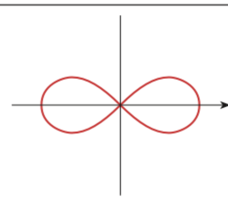
- Analyze the value of r for varying values of θ.

$0 \le t \le \frac{\pi}{4}$	$\frac{\pi}{4} \le t \le \frac{\pi}{2}$	$\frac{\pi}{2} \le t \le \frac{3\pi}{4}$	$\frac{3\pi}{4} \le t \le \pi$

- The resulting curve will have four loops, forming a four-leaved rose.



Common Polar Curves

Circles and Spiral	 $r = a$ circle	 $r = a \sin \theta$ circle	 $r = a \cos \theta$ circle	 $r = a\theta$ spiral
Limaçons $r = a \pm b \sin \theta$ $r = a \pm b \cos \theta$ $(a > 0, b > 0)$ Orientation depends on the trigonometric function (sine or cosine) and the sign of b	 $a < b$ limaçon with inner loop	 $a = b$ cardioid	 $a > b$ dimpled limaçon	 $a \geqslant 2b$ convex limaçon
Roses $r = a \sin n\theta$ $r = a \cos n\theta$ n-leaved if n is odd $2n$-leaved if n is even	 $r = a \cos 2\theta$ four-leaved rose	 $r = a \cos 3\theta$ three-leaved rose	 $r = a \cos 4\theta$ eight-leaved rose	 $r = a \cos 5\theta$ five-leaved rose
Lemniscates Figure-eight-shaped curves	 $r^2 = a^2 \sin 2\theta$ lemniscate	 $r^2 = a^2 \cos 2\theta$ lemniscate		