10.3 Intro to Polar Coordinates

Polar Coordinates and Conversion

Definition (Polar Coordinate System)**.** A point in the plane is represented by *polar coordinates* (r, θ) where:

- r is the distance from the point to a fixed point O (the *pole*).
- θ is the angle (measured in radians) between the polar axis (usually the positive x-axis) and the ray from O to the point.

When $r < 0$, the point (r, θ) is equivalent to $(-r, \theta + \pi)$.

Theorem (Conversion Between Polar and Cartesian Coordinates)**.** If a point has polar coordinates (r, θ) and Cartesian coordinates (x, y), then:

$$x = r\cos\theta, \quad y = r\sin\theta.$$

Conversely, if (x, y) are given, then:

$$r = \sqrt{x^2 + y^2}, \quad \tan\theta = \frac{y}{x}.$$

Properties of Polar Coordinates

Theorem (Multiple Representations)**.** A point can have infinitely many polar representations. In particular,

$$(r, \theta) = \Big(r,\ \theta + 2\pi k\Big) \quad \text{and} \quad (r, \theta) = \Big(-r,\ \theta + (2k+1)\pi\Big)$$

for any integer k.

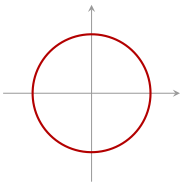
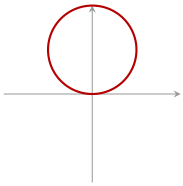
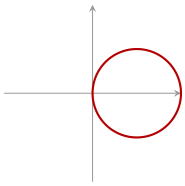
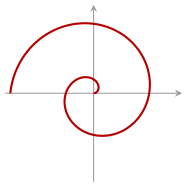
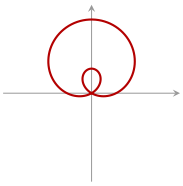
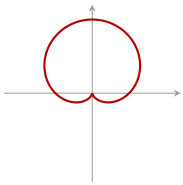
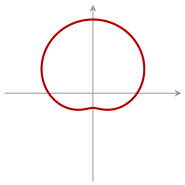
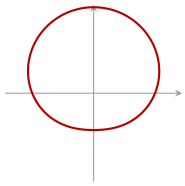
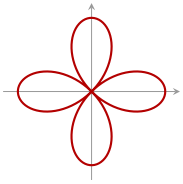
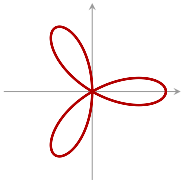
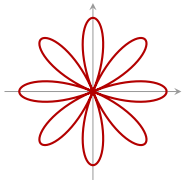
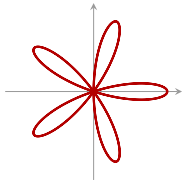
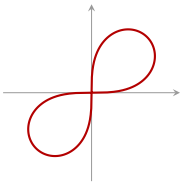
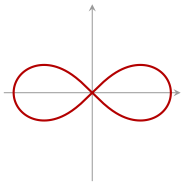
Theorem (Symmetry of Polar Graphs)**.** Let $r = f(\theta)$ be a polar equation. Then:

1. If $f(-\theta) = f(\theta)$ for all θ, the graph is symmetric about the polar axis.
2. If the equation is unchanged when r is replaced by $-r$ (or equivalently when θ is replaced by $\theta + \pi$), then the graph is symmetric about the pole.
3. If $f(\pi - \theta) = f(\theta)$ for all θ, the graph is symmetric about the line $\theta = \frac{\pi}{2}$.

Additional Notes

- A polar curve is defined by an equation of the form $r = f(\theta)$; its graph consists of all points (r, θ) satisfying the equation.
- Many common curves (such as circles, cardioids, limaçons, roses, and lemniscates) have elegant representations in polar coordinates.

Common Polar Graphs

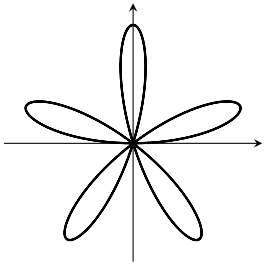
Circles and Spiral	 $r = a$ circle	 $r = a\sin\theta$ circle	 $r = a\cos\theta$ circle	 $r = a\theta$ spiral
Limaçons $r = a \pm b\sin\theta$ $r = a \pm b\cos\theta$ $(a > 0, b > 0)$ Orientation depends on the trigonometric function (sine or cosine) and the sign of b	 $a < b$ limaçon with inner loop	 $a = b$ cardioid	 $a > b$ dimpled limaçon	 $a \geq 2b$ convex limaçon
Roses $r = a\sin n\theta$ $r = a\cos n\theta$ n-leaved if n is odd $2n$-leaved if n is even	 $r = a\cos 2\theta$ four-leaved rose	 $r = a\cos 3\theta$ three-leaved rose	 $r = a\cos 4\theta$ eight-leaved rose	 $r = a\cos 5\theta$ five-leaved rose
Lemniscates Figure-eight-shaped curves	 $r^2 = a^2\sin 2\theta$ lemniscate	 $r^2 = a^2\cos 2\theta$ lemniscate		

Polar Coordinates Problems

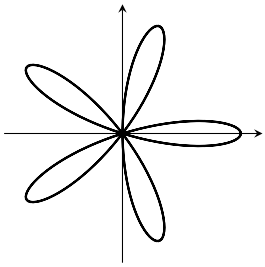
Multiple Choice Matching Problems

For each question, choose the letter (A, B, C, or D) that correctly matches the given description.

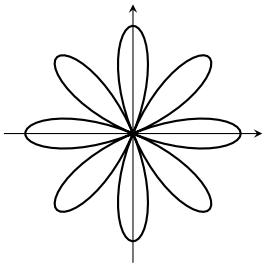
1. Which of the following graphs represents the polar function $r = \sin(5\theta)$?



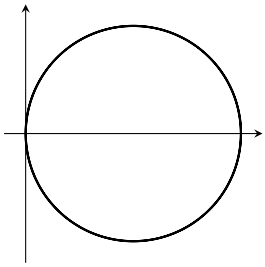
A.



B.

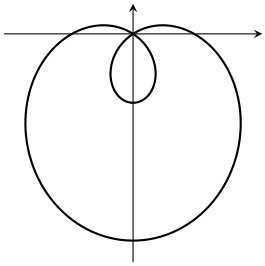


C.

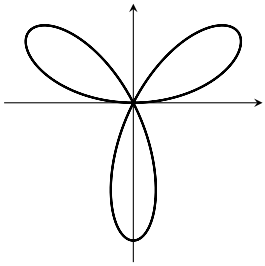


D.

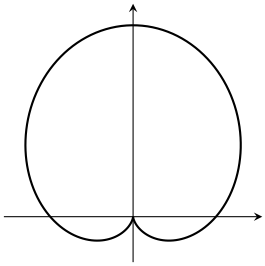
2. Which of the following graphs represents the polar function $r = 1 + \sin\theta$?



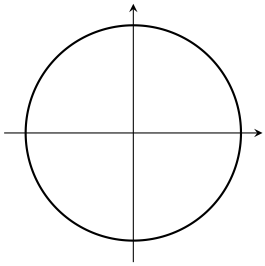
A.



B.

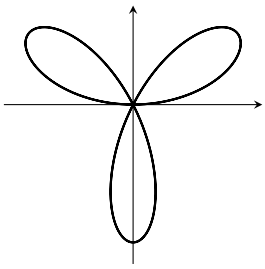


C.

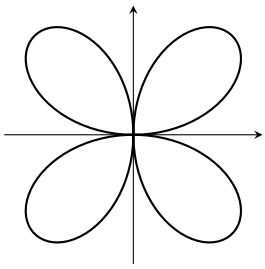


D.

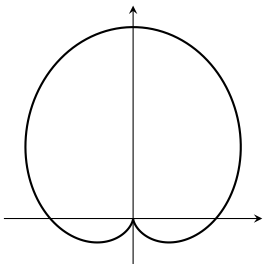
3. Which of the following graphs represents the polar function $r^2 = 2\cos(2\theta)$?



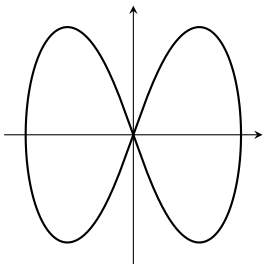
A.



B.

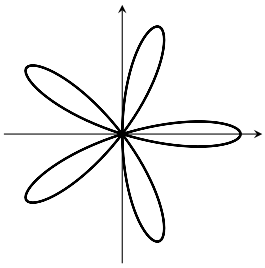


C.

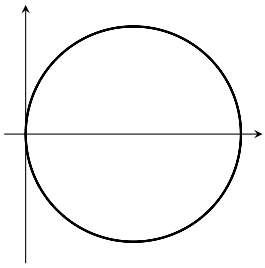


D.

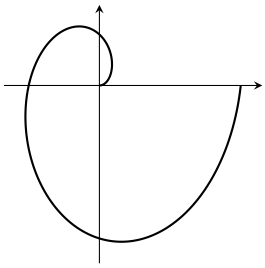
4. Which of the following graphs represents the polar function $r = 2\cos\theta$?



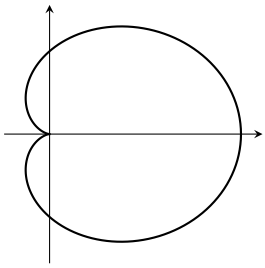
A.



B.

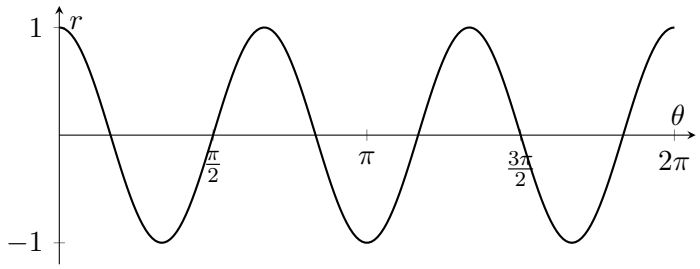


C.

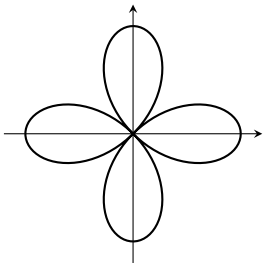


D.

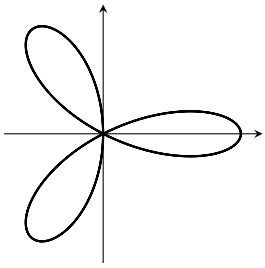
5. The following figure shows a graph of r as a function of θ.



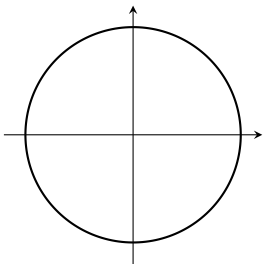
Which of the following corresponds to the curve on the Cartesian xy-plane, using polar coordinates?



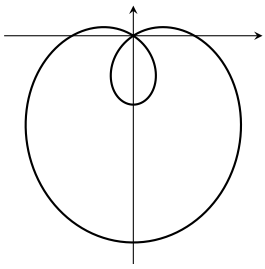
A.



B.

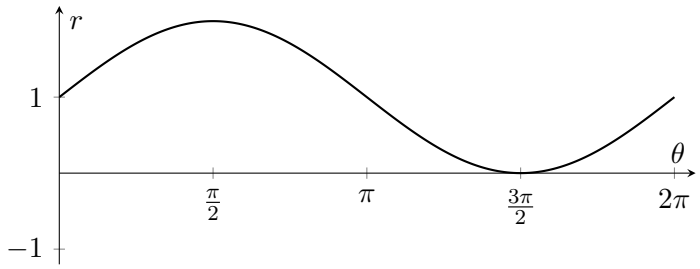


C.

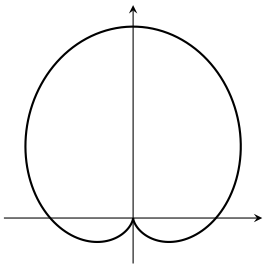


D.

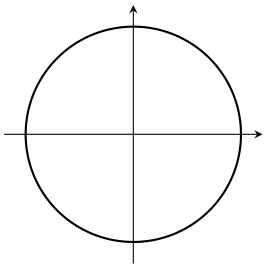
6. The following figure shows a graph of r as a function of θ.



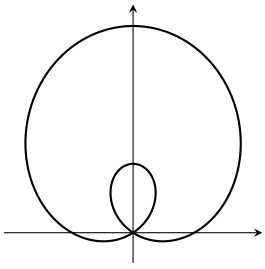
Which of the following corresponds to the curve on the Cartesian xy-plane, using polar coordinates?



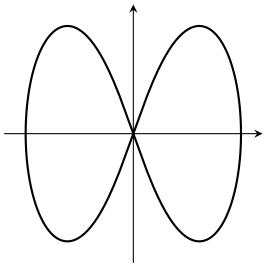
A.



B.

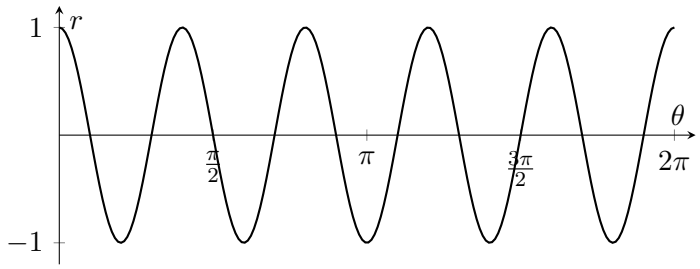


C.

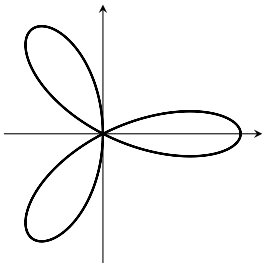


D.

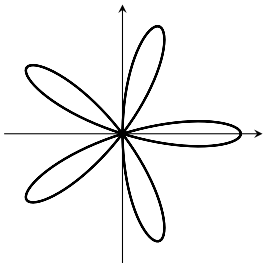
7. The following figure shows a graph of r as a function of θ.



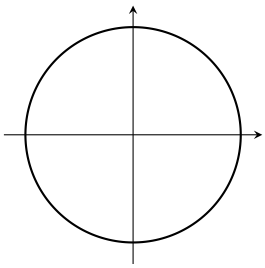
Which of the following corresponds to the curve on the Cartesian xy-plane, using polar coordinates?



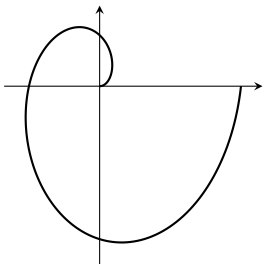
A.



B.

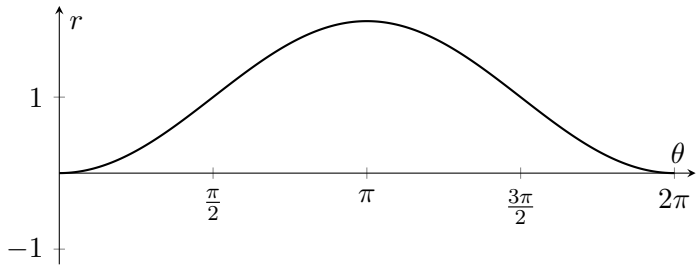


C.

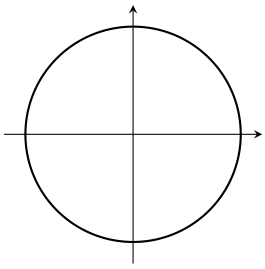


D.

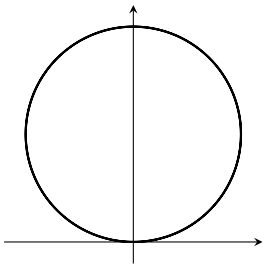
8. The following figure shows a graph of r as a function of θ.



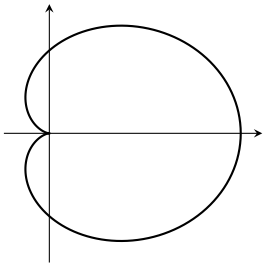
Which of the following corresponds to the curve on the Cartesian xy-plane, using polar coordinates?



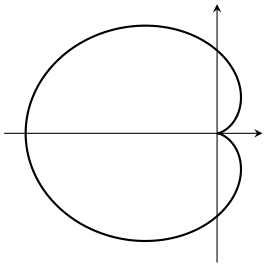
A.



B.



C.



D.